FLUIDIC scaling in MEMS

H. Shea

v. 2024

Fluidics scaling: table of contents

1. Intro: “Life at low Reynolds number” (key paper by E. Purcell)

2. Reynolds number
3. Drag and sedimentation
4. Flow profile in small channels
5. Forces on particles in liquid flow
6. Diffusion and Mixing
7. Knudsen number and low-pressure regime
8. Surface Tension and capillary pressure

T. M. Squires and S. R. Quake: Microfluidics:
Fluid physics at the nanoliter scale, Rev Mod Phys, 2005

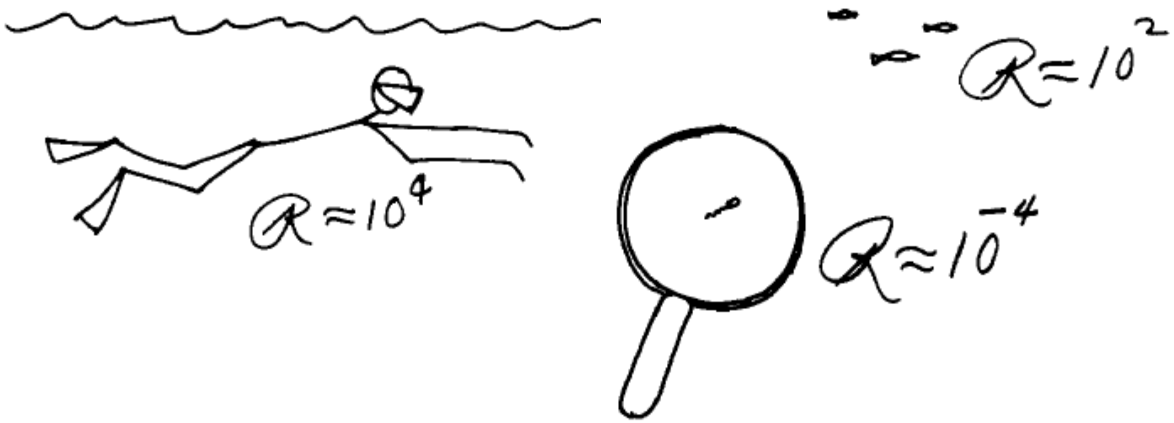
Key fluidic concepts

- Reynolds number *Re*.
    - How do fluids flow when $Re < 10$? what is different from flow at higher *Re*?
    - Drag and sedimentation of particles in a liquid
    - Flow in a circular and rectangular channel of multiple liquids - sheath flow
- How do particles behave in a laminar flow. Near walls?
- Diffusion in liquids: scaling? When is diffusion relevant
- Mixing by diffusion: different ways this can be exploited or accelerated for laminar flows
- Knudsen number: rarified gas regime
    - Damping, Q: different regimes
    - Squeeze film damping: what phenomena take place?
- Surface tension
    - Scaling of capillary pressure, Washburn eq, capillary stop valves
    - Electrowetting
    - Droplet formation

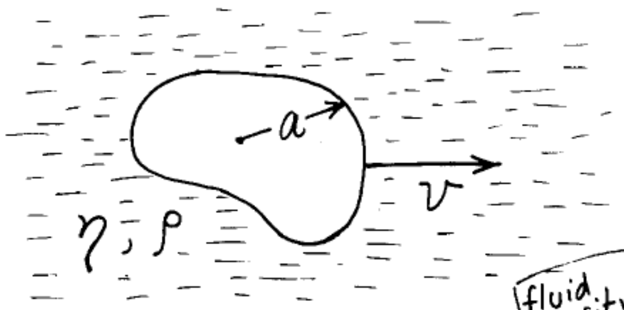
Life at Low Reynolds Number (1976) E.M. Purcell

Edward Mills Purcell

- Professor of Physics at Harvard
- Nobel Prize winner for NMR



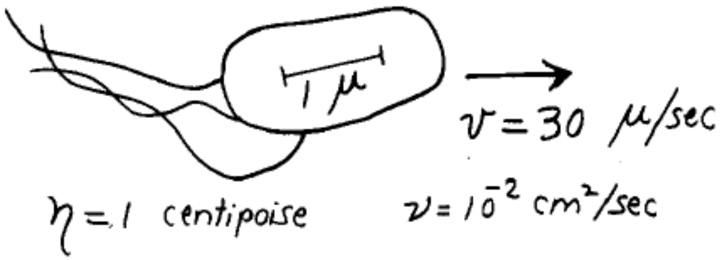
Reynolds Number ($\Re$ or Re)



fluid density

$$\mathcal{R} = \frac{\text{inertial forces}}{\text{viscous forces}} \approx \frac{a v \rho}{\eta}$$

fluid viscosity



Re=10^{-5}: "human in a pool of molasses with arms moving at 1 cm/minute)"
(Purcell, 1997)

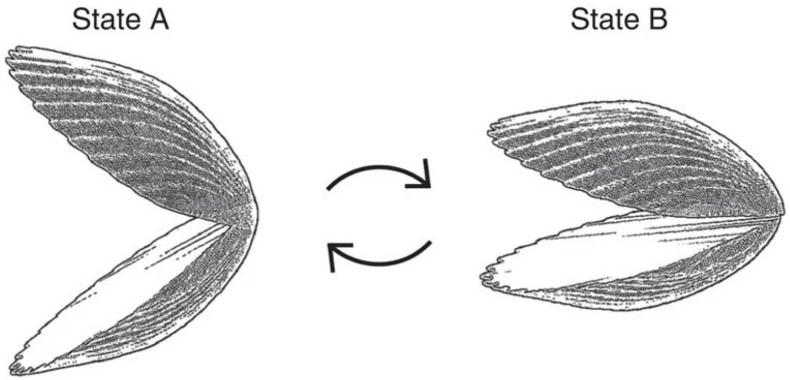
$$\mathcal{R} = 3 \times 10^{-5}$$

$$\left\{ \begin{array}{l} \text{coasting distance} = 0.1\ \text{Å} \\ \text{coasting time} = 0.3\ \text{microsec.} \end{array} \right\}$$

- Low Re = laminar flow = viscous forces dominate
- High Re = turbulent flow = inertia dominates

E. M. Purcell, Life at low Reynolds number. *American Journal of Physics* **45**, 3–3 (1977).

Simple micro-swimmers

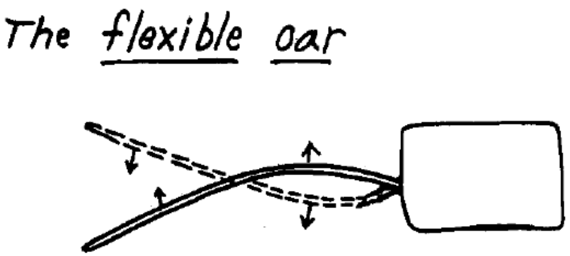


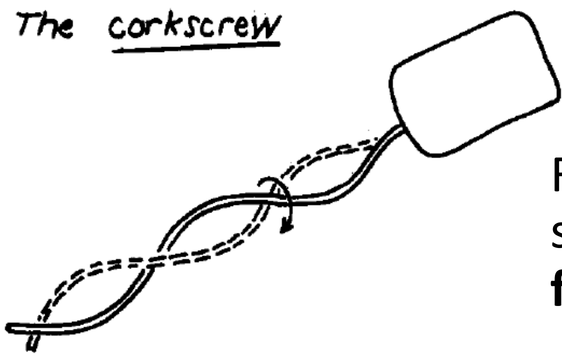
The Scallop Theorem

Reciprocal motion => at low Re, will return to original position

→ Closing faster than you open doesn’t work

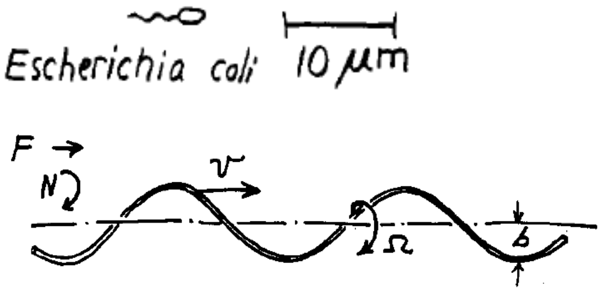
Solutions





Real-world micro-swimmers use **flagella**

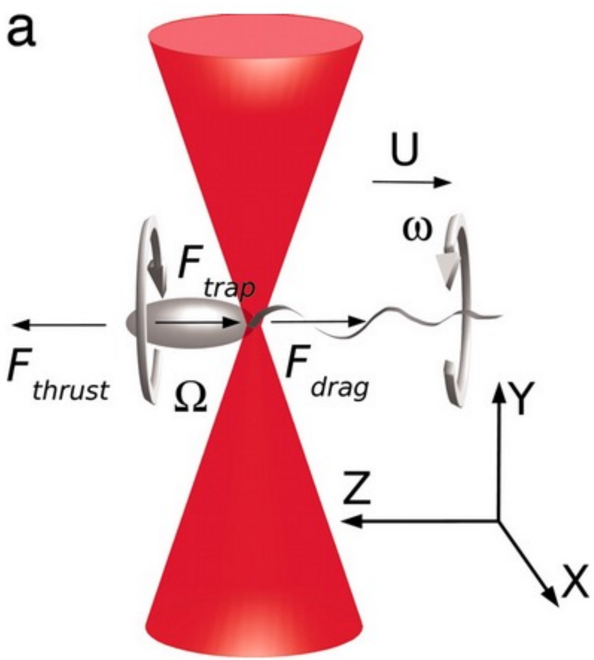
Efficiency of micro-swimmers



Propulsive efficiency $\propto B^2$

$$B \propto \left(\frac{\text{transverse drag}}{\text{longitudinal drag}} - 1 \right)$$

Optically trap *E. coli*



(Chattopadhyay, et al., 2006)

Energy required, if efficiency of propulsion is 1%:

2×10^{-8} erg/sec,

or $\frac{1}{2}$ watt/kilogram

Human on a bike = **~5 W/kg**

Car = **~100 W/kg**

Theoretical efficiency = **~1%**

Experimental efficiency = **~2%**

The significance of diffusion

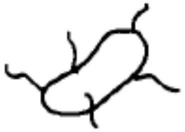
0.5 W/kg ➔ diffusion is enough to feed if density of [energetic molecules] = **10^{-9} M**

Stirring vs. Diffusion

➔ **Sherwood number** (**S**) = $\frac{t_{diffusion}}{t_{stirring}}$

"$\mathcal{S}$" $= \frac{\ell v}{D}$ $\quad \left[\mathcal{R} = \frac{\ell v}{\nu} \right]$

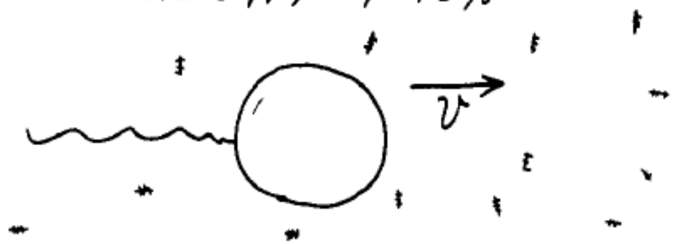
$D < 10^{-5}\ cm^2/sec$ $\quad$ $\nu < 10^{-2}\ cm^2/sec$



$\mathcal{S} \lesssim 10^{-2}$

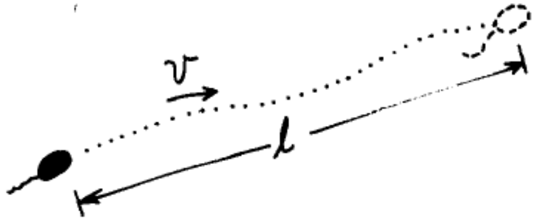
local stirring accomplishes nothing

to increase supply by 10% :



$v = 1.4\, D/a = \boxed{700\ \mu/sec}$

Actual top speed = ~**35 µm/s**



to out-swim diffusion:

$\ell \geq D/v$

if $D = 10^{-5}\ cm^2/sec$, $v = .003\ cm/sec$

$\ell \gtrsim 30\ \mu$

"If you don't swim that far you haven't gone anywhere."

Table 2. Time required for diffusion of O_2 over a range of distances.

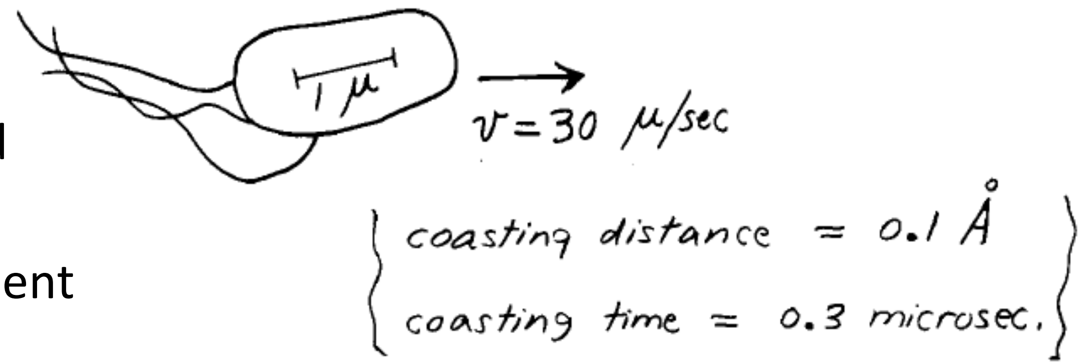
Distance of Diffusion	Approximate Time Required
10 nm	23.8 ns
50 nm	595 ns
100 nm	2.38 µs
1 µm	238 µs
10 µm	23.8 ms
100 µm	2.38 s
1 mm	3.97 min
1 cm	6.61 hours
10 cm	27.56 days

$$t \approx \frac{x^2}{2D}$$

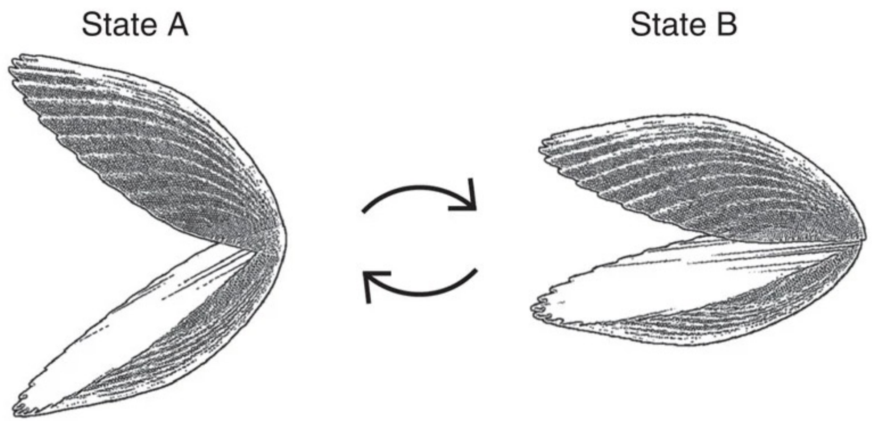
Summary – Life at Low Re

1. In low Re situations, inertia is irrelevant and viscosity dominates

→ Only the forces present at the moment matter—the past doesn't matter



2. Microorganisms swim by different rules – low efficiency, cyclic configurations



3. Diffusion is powerful: need to swim a minimum distance to benefit

$$t_{diffusion} \propto x^2$$

2. Reynolds number *Re*

The Reynolds number *Re*

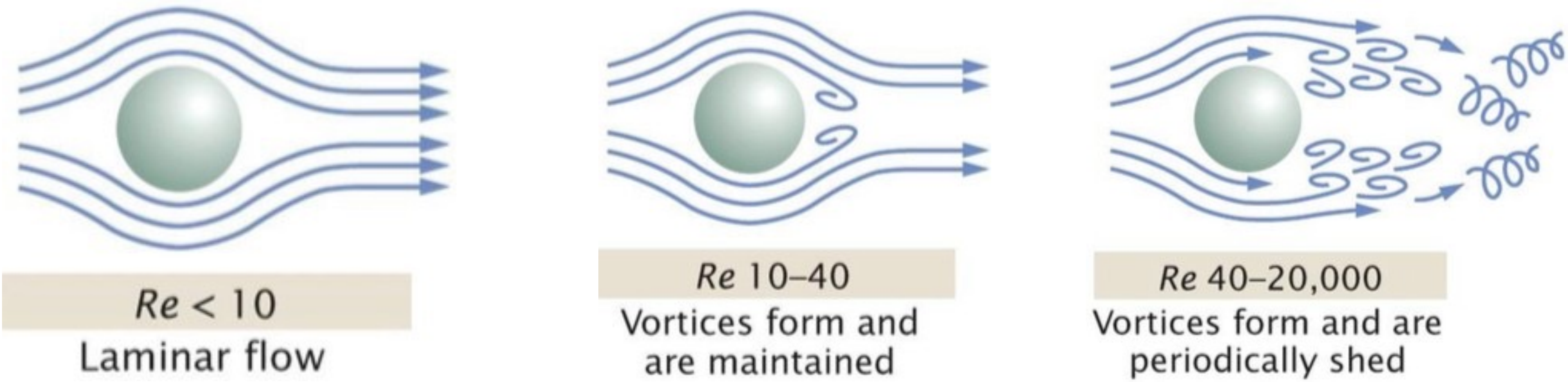
- ρ : density [kg m^{-3}]
- *D* : typical size [m]
- v : speed [m s^{-1}]
- η : dynamic viscosity [Pa.s]

Reynolds number $$\boxed{Re = \frac{\rho \bar{v} D}{\eta} \propto \, v\, L}$$ $$\frac{inertial\ forces}{viscous\ forces}$$

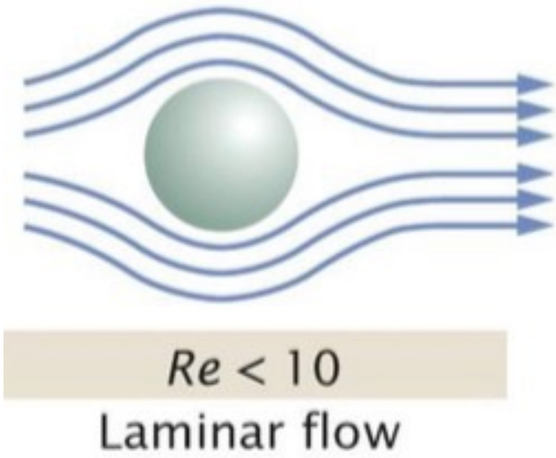
<u>Microfluidics</u>: *D* and *v* are small → small Reynolds number → laminar flow

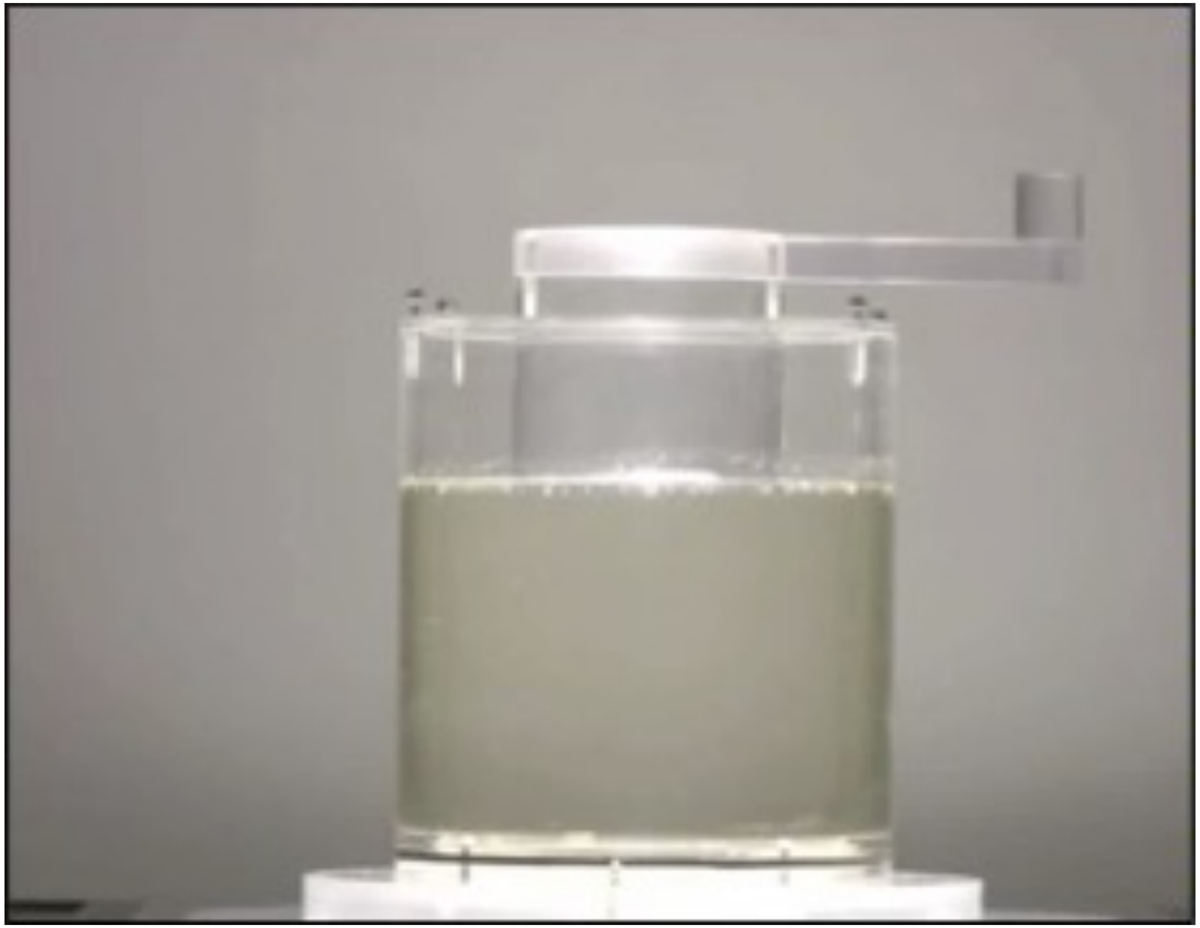
<u>*Numerical*</u>*:* channel dia. 20 µm, v = 1 mm/s, water (η=10^{-3} Pa.s) → *Re* = 0.04

- Small Re: No turbulence, but stable vortices are possible
- Stokes equation: <u>reversibility</u> in time (… if no diffusion or chemical reactions take place)



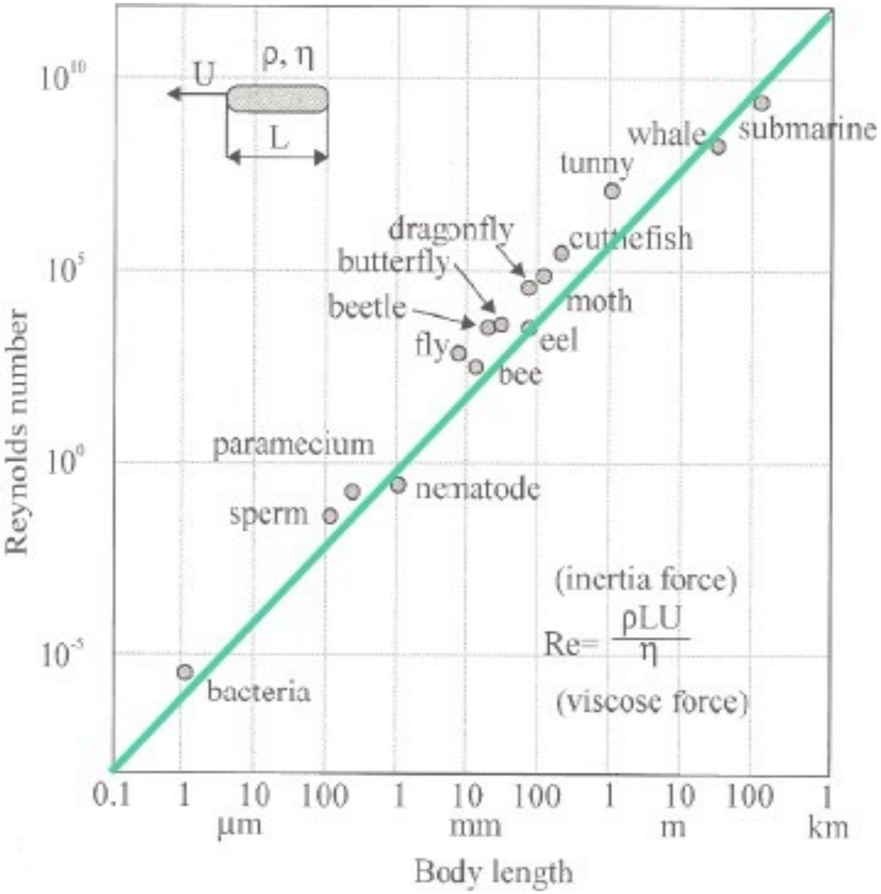
chaotic

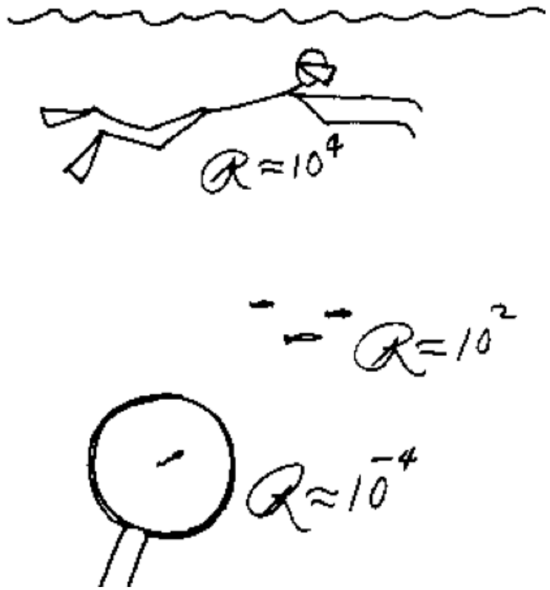


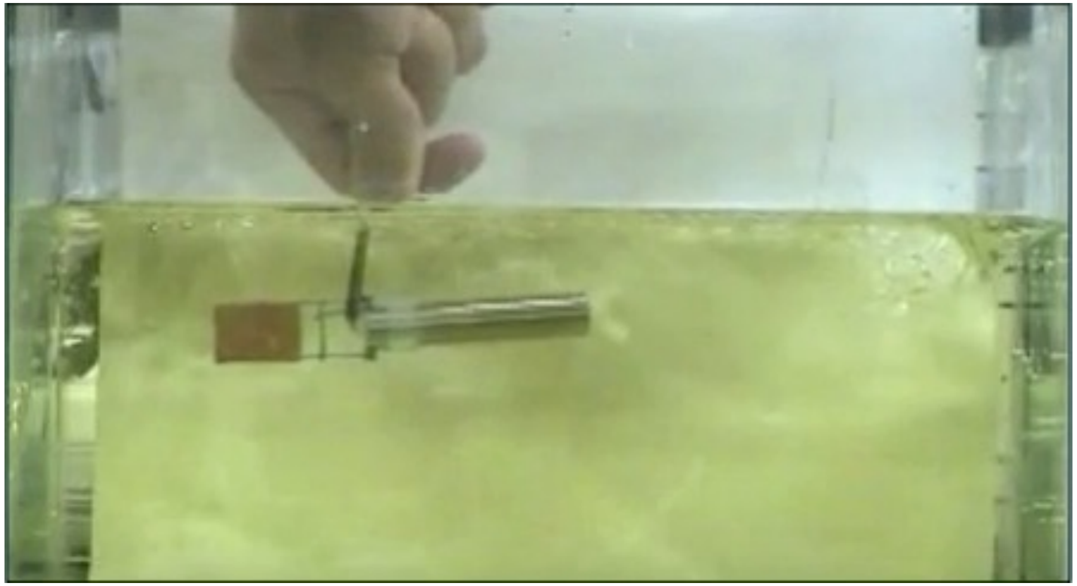


when the flow is laminar: reversibility in time
(when low/no diffusion)

Reynolds number of objects and animals







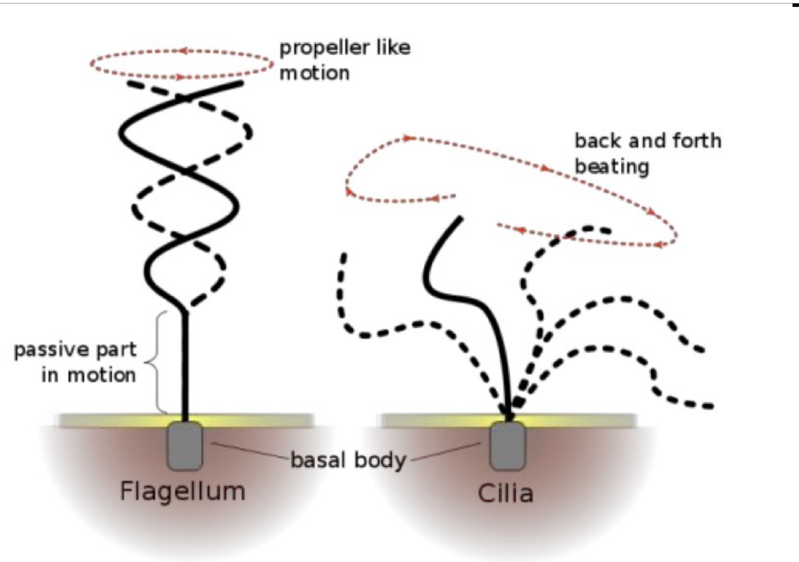
- *«Micromachines: A New Era in Mechanical Engineering" by Iwao Fujimasa, Oxford University Press, 1996*
- *"Life at Low Reynolds Number", E.M. Purcell, 1976*

Bacteria swimming in water:

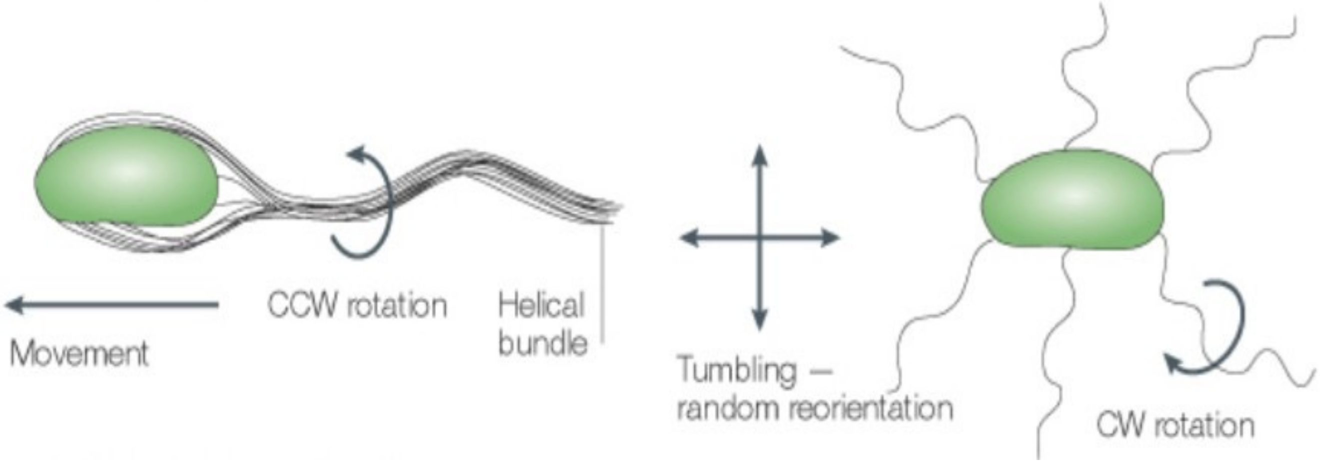
Speed ~ 30 x 10^{-6} m/s ; size ~ 1 x 10^{-6} m

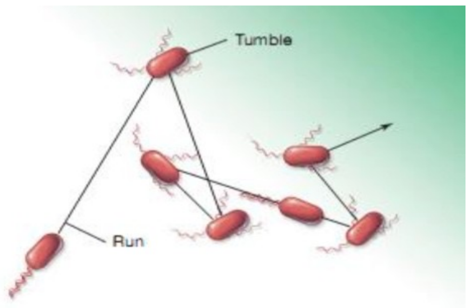
$Re \sim 1\ x\ 10^{-5}$ = laminar

So bacteria swimming in water is like us in honey

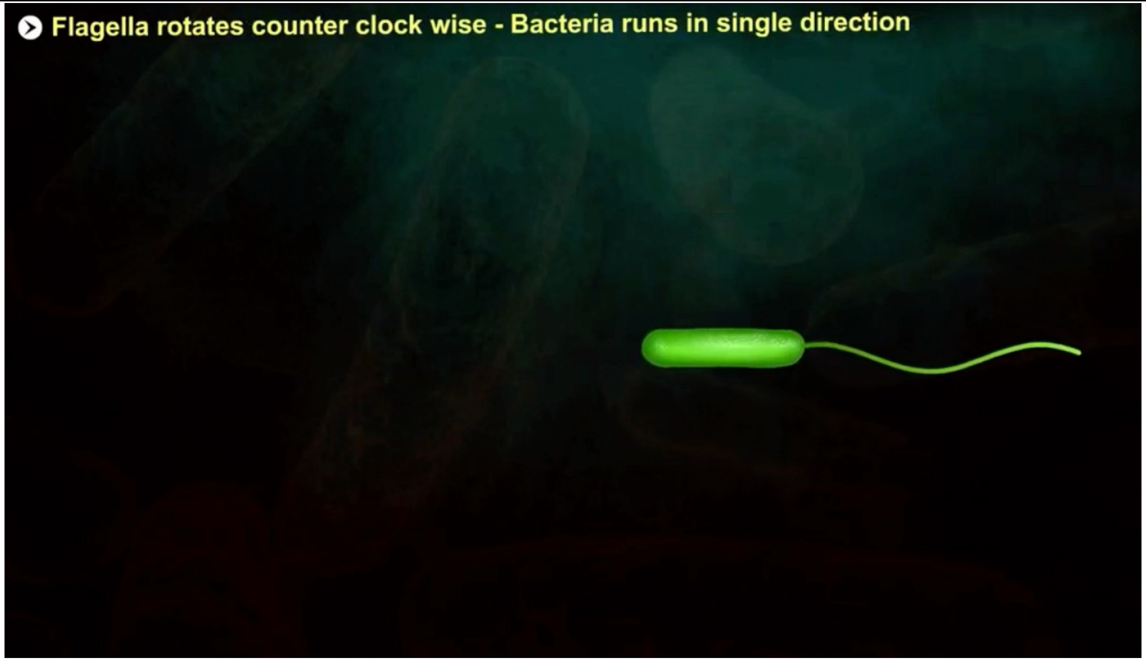


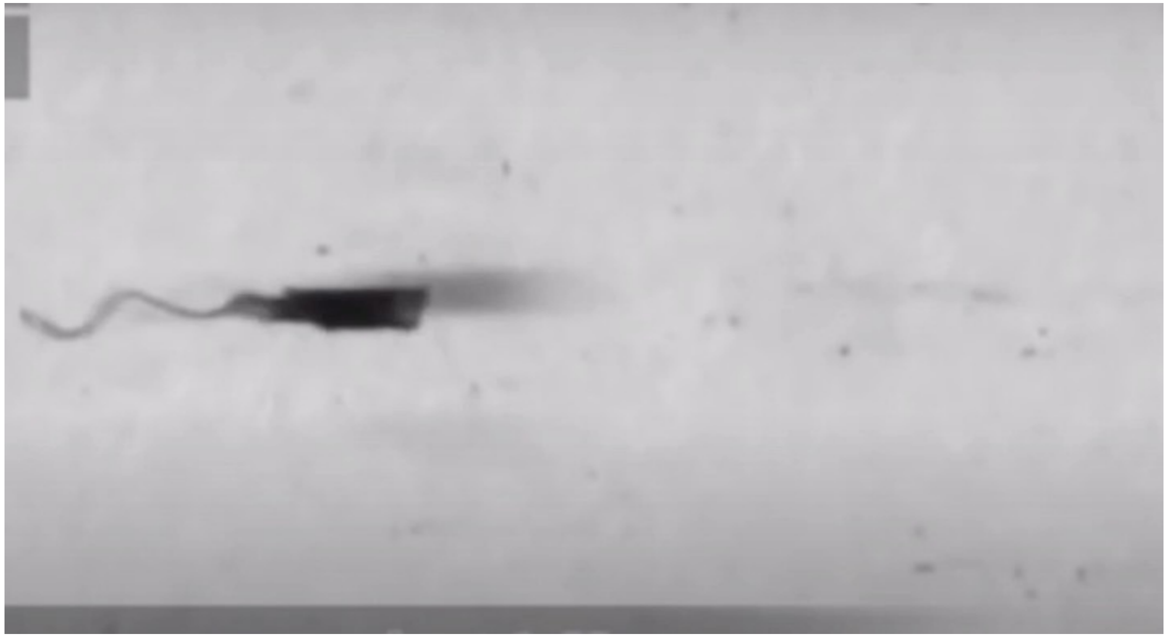
Flagellum and cilia. The former has a rotatory motion while the latter move from side to side (from Wikipedia http://en.wikipedia.org/wiki/Flagellum).





Swimming





Selman Sakar, EPFL
Magnetic actuation

3. Viscous drag and particle sedimentation

Viscous drag on objects in liquids

$$F_{stoke} = 6\pi\, R\, \eta\, v(t)$$

Stoke's drag for sphere radius R in a liquid
η : Dynamic viscosity [Pa.s]

Stop distance x_0 of small objects in viscous media (*low Re*)

$$m\frac{dv(t)}{dt} = 6\pi\, R\, \eta\, v(t)$$

$$v(t) = v_0 e^{-t/\tau} \quad \text{where} \quad \tau = \frac{\rho R^2}{3\eta}$$

=> stop distance $x_0 = \dfrac{v_0 \rho R^2}{3\eta}$ $\quad \propto R^2 \propto L^2$

Numerical application: a particle of radius *r*, moving at a speed of 10 times its radius per second, in water, particle has the same density as water to ignore buoyancy

r = 10 μm, v=100 μm/s => $x_0 = \dfrac{10^{-4} \cdot 10^{3} 10^{-10}}{3 \cdot 10^{-3}} = 3\ nm$ **Re = 0.00002**

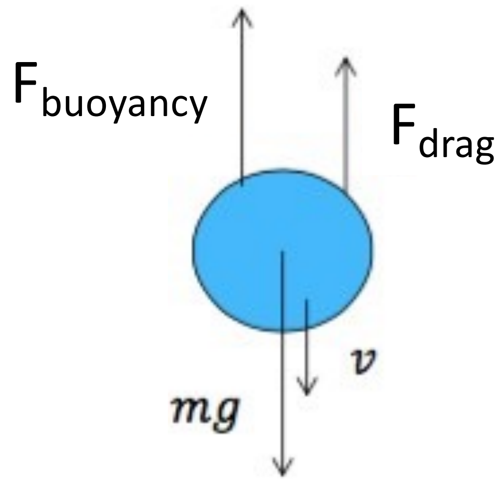
r = 1 mm, v=10 mm/s => $x_0 = \dfrac{10^{-2} \cdot 10^{3} 10^{-6}}{3 \cdot 10^{-3}} = 3\ mm$ Re = 0.2

r=33 m, v=5 m/s x_0= 10^6 km ! Re= 3.10^8. so not applicable!

Sedimentation of particles

Equilibrium speed v_{sed}: drag force = weight

$$6\pi r\eta\, v_{sed} = \Delta\rho \cdot V \cdot g$$



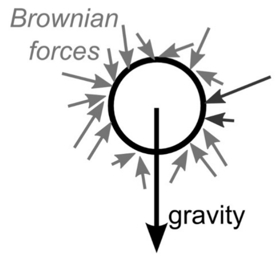
Sedimentation speed $v_{sed} = \frac{\Delta\rho V g}{6\pi r\eta} = \frac{2}{9}\frac{\Delta\rho g}{\eta} r^2$

$$\boxed{v_{sed} \propto r^2 \propto L^2}$$

Small objects settle more slowly

Size limit for particles in suspension: diffusion (Brownian)

The diffusional force exerted on a particle can be approximated by $F_{diff} \approx \frac{k_B T}{2r} \propto L^{-1}$

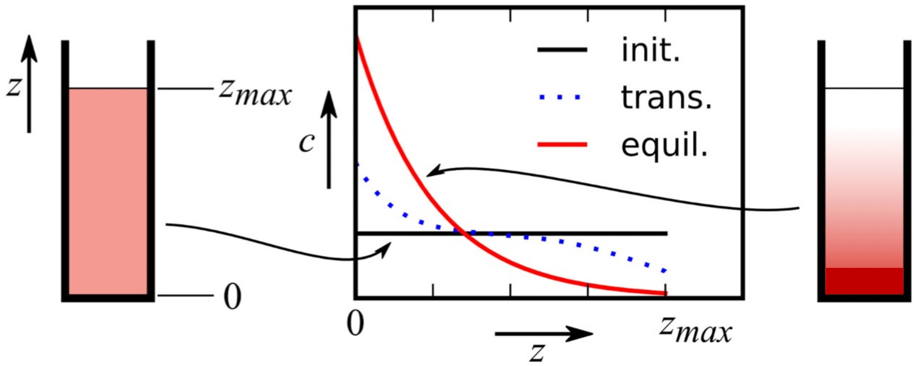


The particle sediments **much more slowly** when the Brownian motion force is approx equal to buoyancy force, because of diffusional broadening.

Brownian forces dominate when $\frac{k_B T}{2r} > \frac{4}{3}\pi r^3 \Delta\rho\, g$ i.e. when $r^4 < \frac{3 k_B T}{8\pi\,\Delta\rho\, g}$

r_{crit} approx **2 µm** diameter for a particle of density 2000 kg/m^3

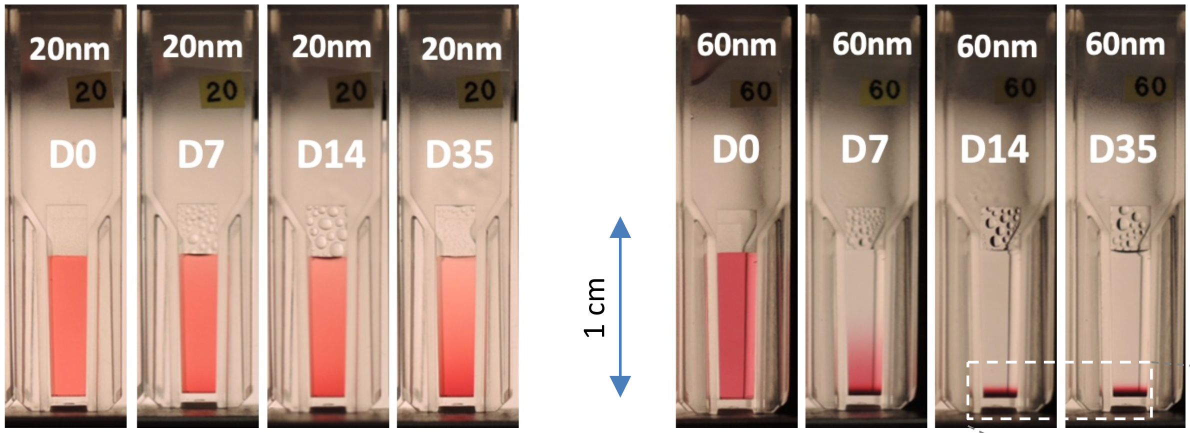
Sedimentation of particles when dominated by Brownian motion



$$\frac{k_B T}{2r} > \frac{4}{3}\pi r^3 \Delta \rho g$$

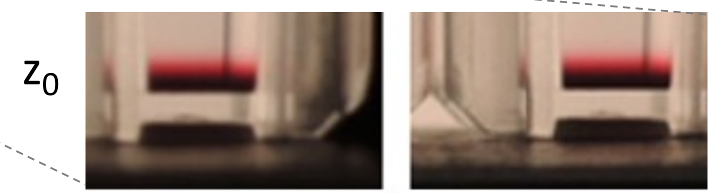
characteristic height of the equilibrium gradient, $z_0 = \dfrac{k_\mathrm{B}T}{m_b g}$

Gold nanoparticles, 35 days of observation



20nm = 1cm / 45days
z_0 = 5 mm

60nm = 1cm / 5days
z_0 = 0.18 mm



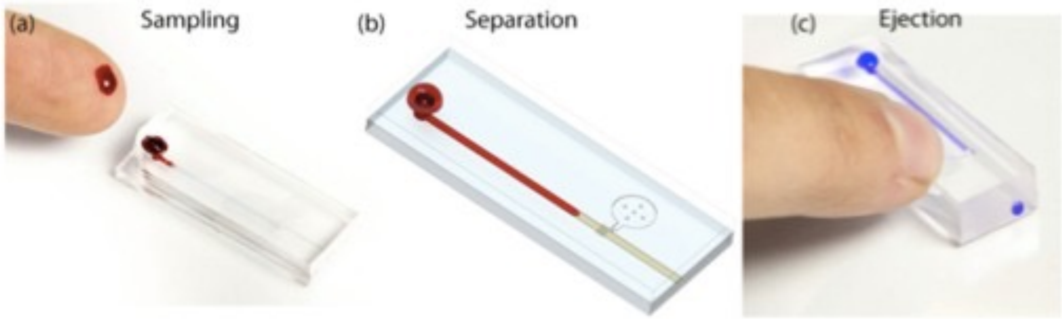
Gold nanospheres in water

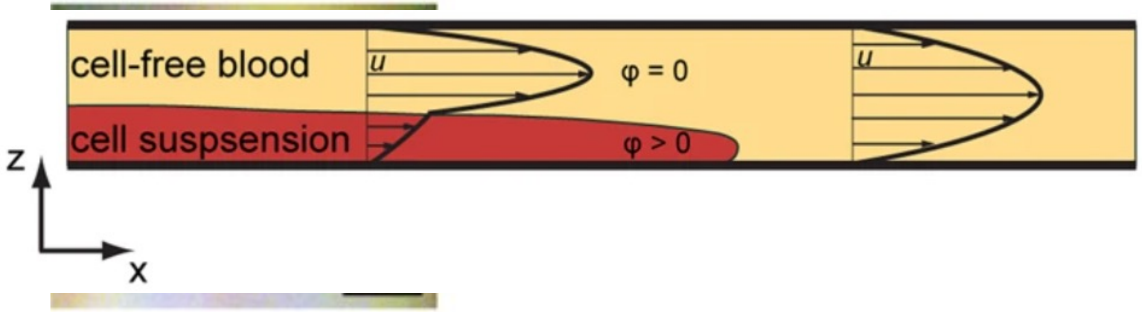
diam. nm	D $\mu m^2 s^{-1}$	s 10^{-9} s	z_0 mm	t^{1cm}_{sed} hours
13	19.9	0.110	18.5	2579
20	13.0	0.260	5.08	1090
40	6.48	1.04	0.636	272
50	5.18	1.62	0.325	174
60	4.32	2.34	0.188	121

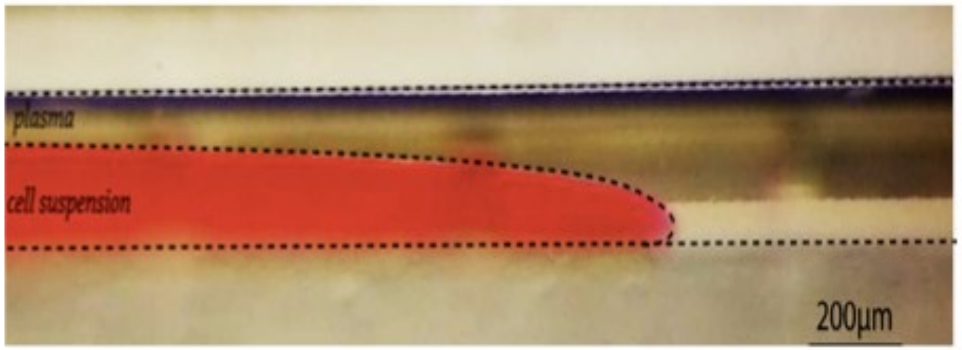
Midelet et al, Part. Part. Syst. Charact. 2017 https://doi.org/10.1002/ppsc.201700095

Plasma extraction by local sedimentation

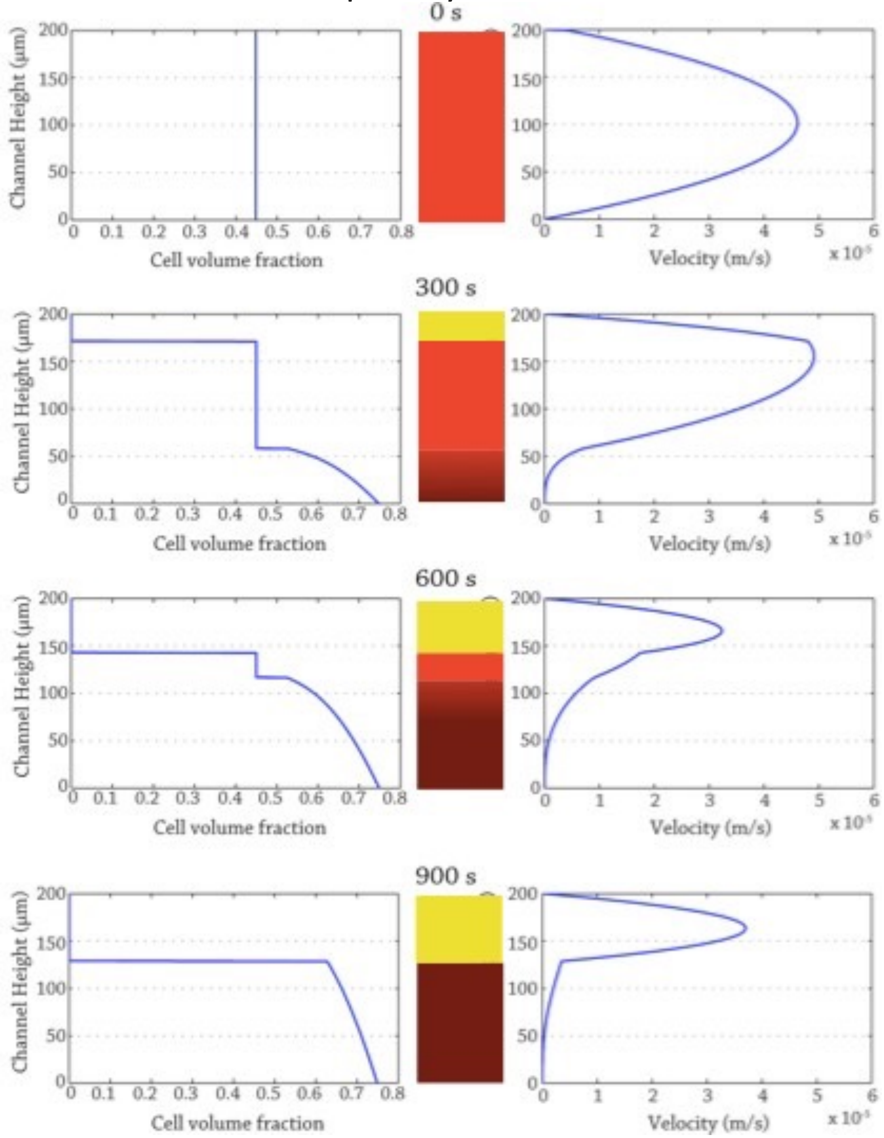
capillary-driven microfluidic device that separates blood microsamples collected from finger-pricks and delivers 2 µL of metered serum for bench-top analysis







D. Forchelet, PhD thesis EPFL 2017



D. Forchelet et al, Sci. Rep. 2018

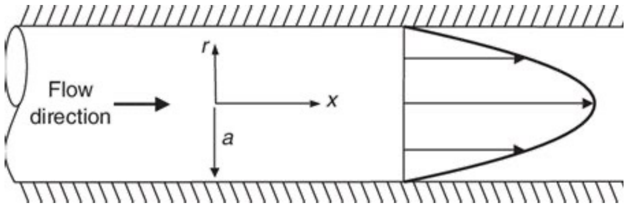
Doi: 10.1038/s41598-018-32314-4

4. Flow profile in small channels

Pressure drop and flowrate

For a circular cross-section, assuming

- Parabolic velocity profile
- zero flow at the wall



Length l
Radius r
Viscosity η

average Fluid velocity $v = \dfrac{r^2}{8\eta l}\Delta p$ $\qquad v \propto \dfrac{r^2}{l}$ $\qquad \propto L$

Pressure drop $\Delta p = \dfrac{8\eta l}{r^2}\bar{v}$ $\qquad \dfrac{\Delta p}{\Delta x} \propto \dfrac{1}{r^2}$ $\qquad \propto L^{-2}$

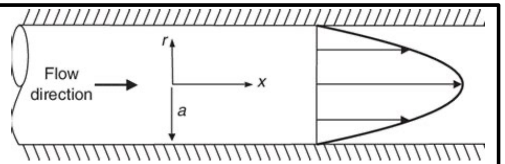
Flow rate $Q = \pi r^2 \bar{v} = \dfrac{\pi r^4}{8\eta l}\Delta p$ $\qquad Q \propto \dfrac{r^4}{l}$ $\qquad \propto L^3$

eg Channel: diameter $2r$=20 µm , l=10 mm , water

v = 0.1 mm/s	=>	Δp=0.8 mbar	Q = 0.03 nl/s
Δp=100 mbar	=>	v=12.5 mm/s	Q = 66 nl/s

Microfluidics are only suited to small volumes…

Flow profile in a rectangular channel as we often have in microfluidics



If w: width >> h: height : i) parabolic profile in y direction, ii) flat velocity profile in x

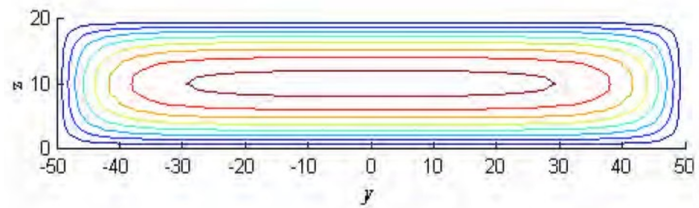
$$\Delta p = \frac{12\eta l}{h^2}\bar{v} \qquad \Delta p \propto \frac{1}{h^2}$$

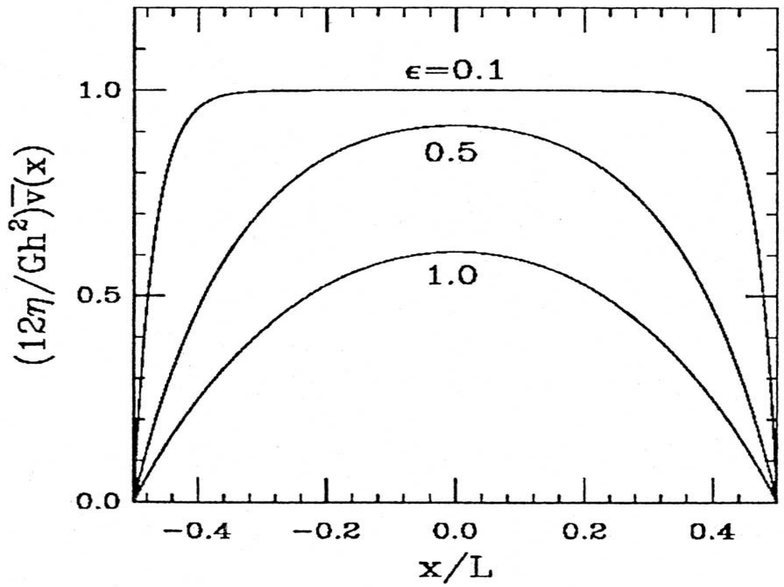
Flow rate $$Q = w \cdot h \cdot \bar{v} = \frac{w \cdot h^3}{12\eta l}\Delta p$$

Fluidic resistance $$R_f = \frac{12\eta l}{wh^3} \propto L^{-3}$$

For w/h ratio not too large (eg 5): non-flat profile in x

$$Q \approx \frac{h^3 w \Delta p}{12\eta L}\left(1 - 0.630\frac{h}{w}\right)$$





Numerical: example

Water at Δp=1 bar

h= 50 nm , w=2 µm, l=10µm ,

Then v= 1 mm/s

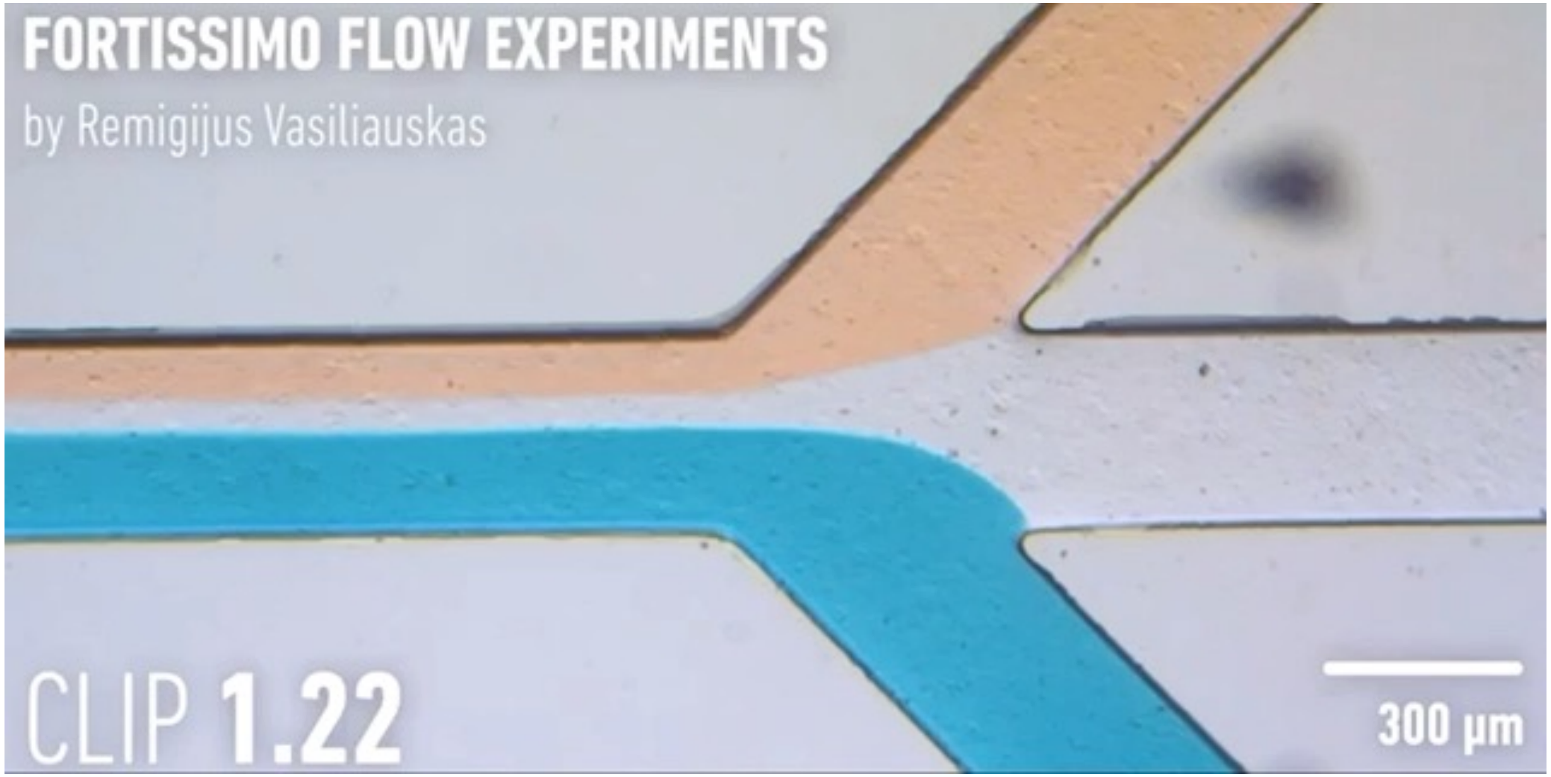
Q= 0.05 nl/s

https://www.youtube.com/watch?v=P09fuC1POwM

Sheath flow

- To create narrow flow line
- To focus stream (and particles)

Inlet 1



Inlet 2

Inlet 3

flow

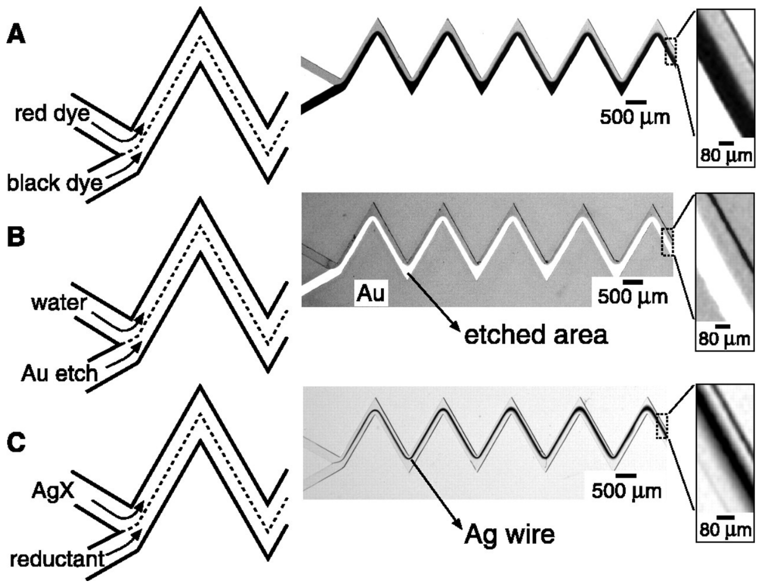
Varying the relative flow rates of the three inlets

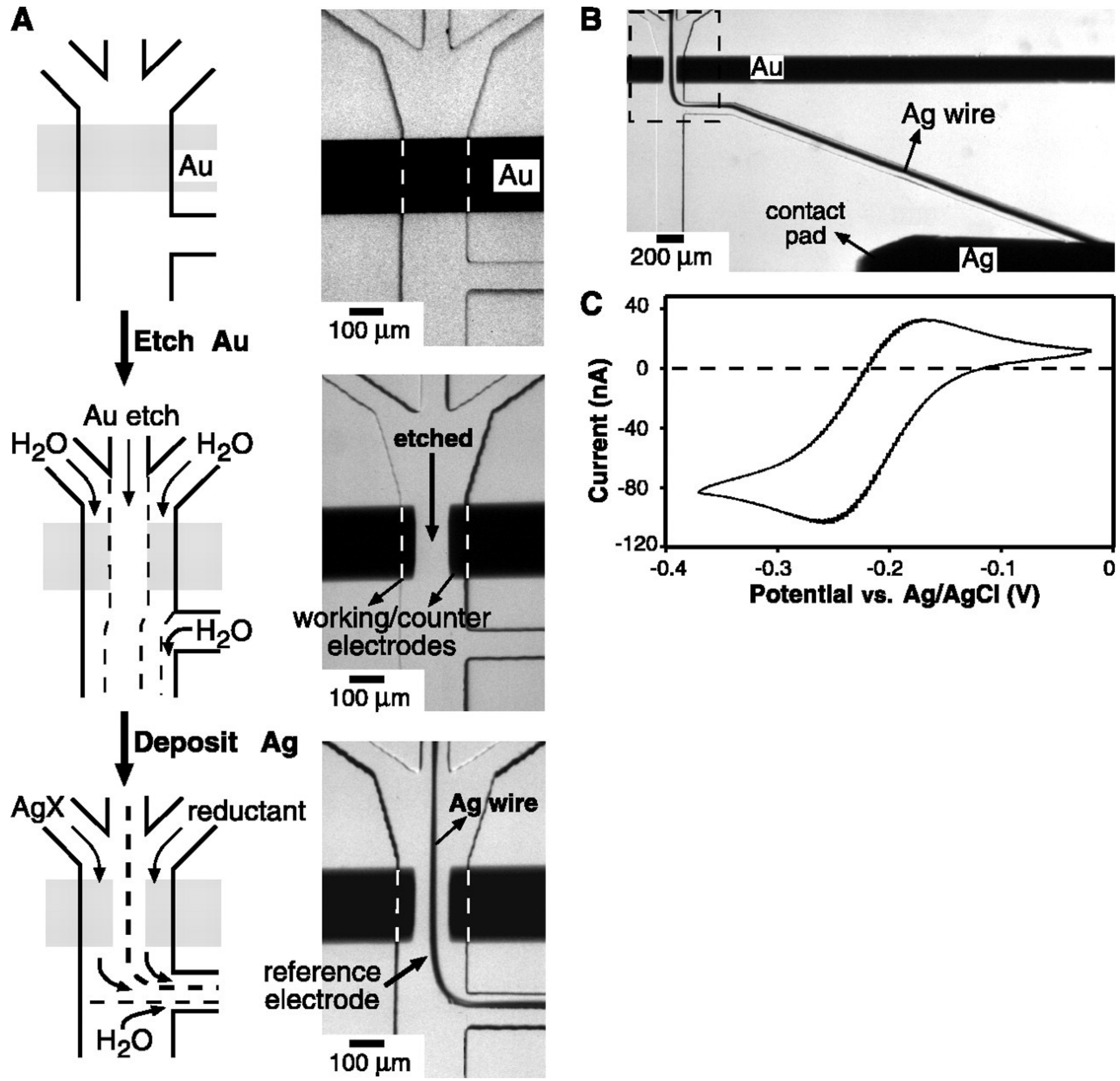
Particles will stay with flow lines of the liquid

Sheath flow: an example application

<u>Sheath flow</u>

- To create narrow flow line
- To focus stream (and particles)





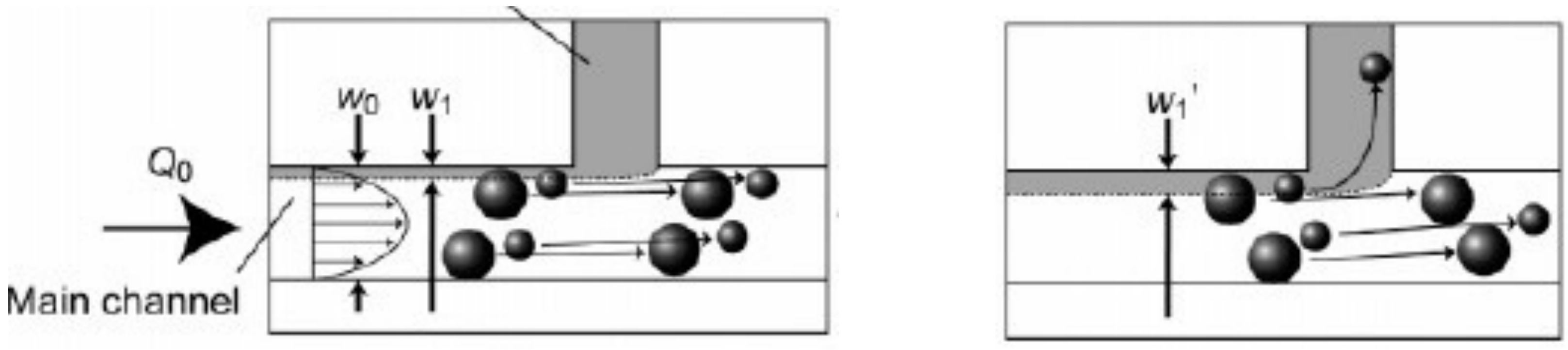
Kenis et al, Science 1999

5. Particles in laminar flows

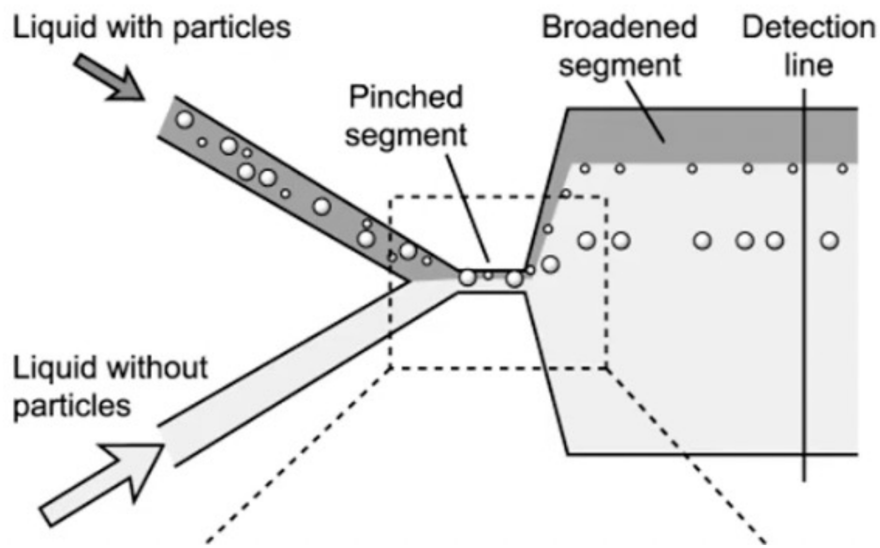
Particles in laminar flows

Filters that never clog: When the size of particles becomes commensurable with the channel size, the particles can be deviated from their flux line by contact with a wall or an obstacle.

In the example below, only small particles will be extracted. Tune size by tuning flow rates



Yamada et al, Anal. Chem. **2006,** 78,1357-1362

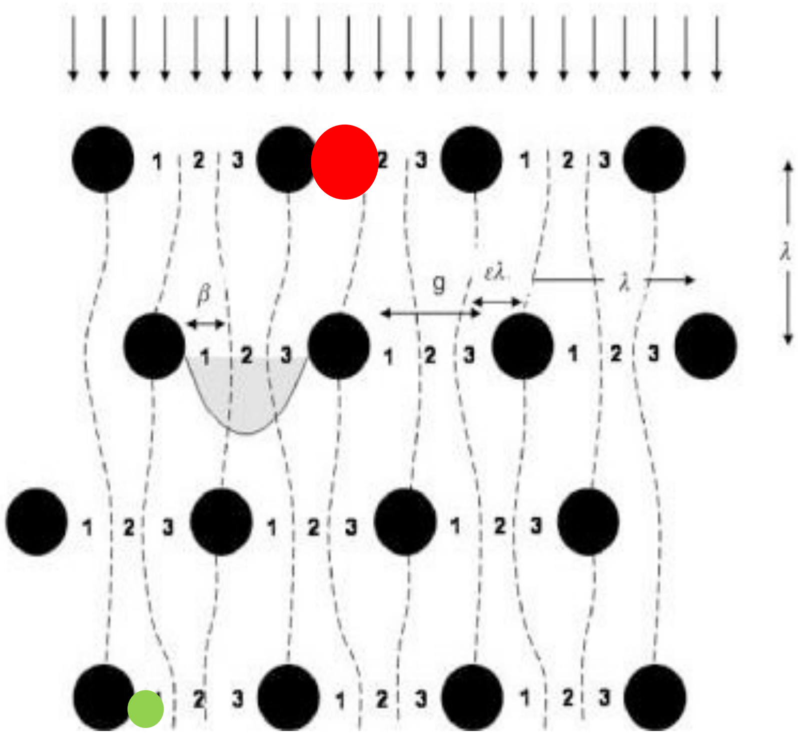


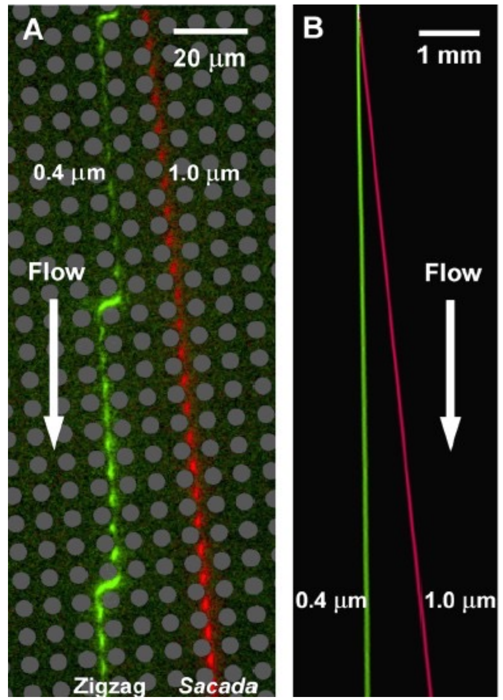
Pinched flow fractionation (PFF): By merging two flows, it is also possible to push particles against the wall and so use this effect for separating particles by size

M Yamada, et al,
Analytical Chemistry **2004** *76* (18), 5465-5471
DOI: 10.1021/ac049863r

Particle in laminar flows – Tango array: sort particles by size

By placing obstacles is a flow with a lateral shift in each row, it is possible to sort particles by size in an efficient manner. In the following example, the displacement is 1/3 of obstacle spacing.

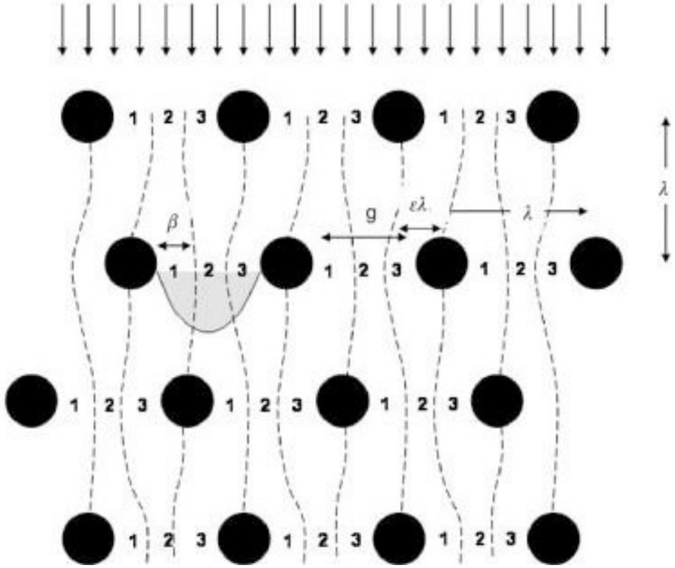


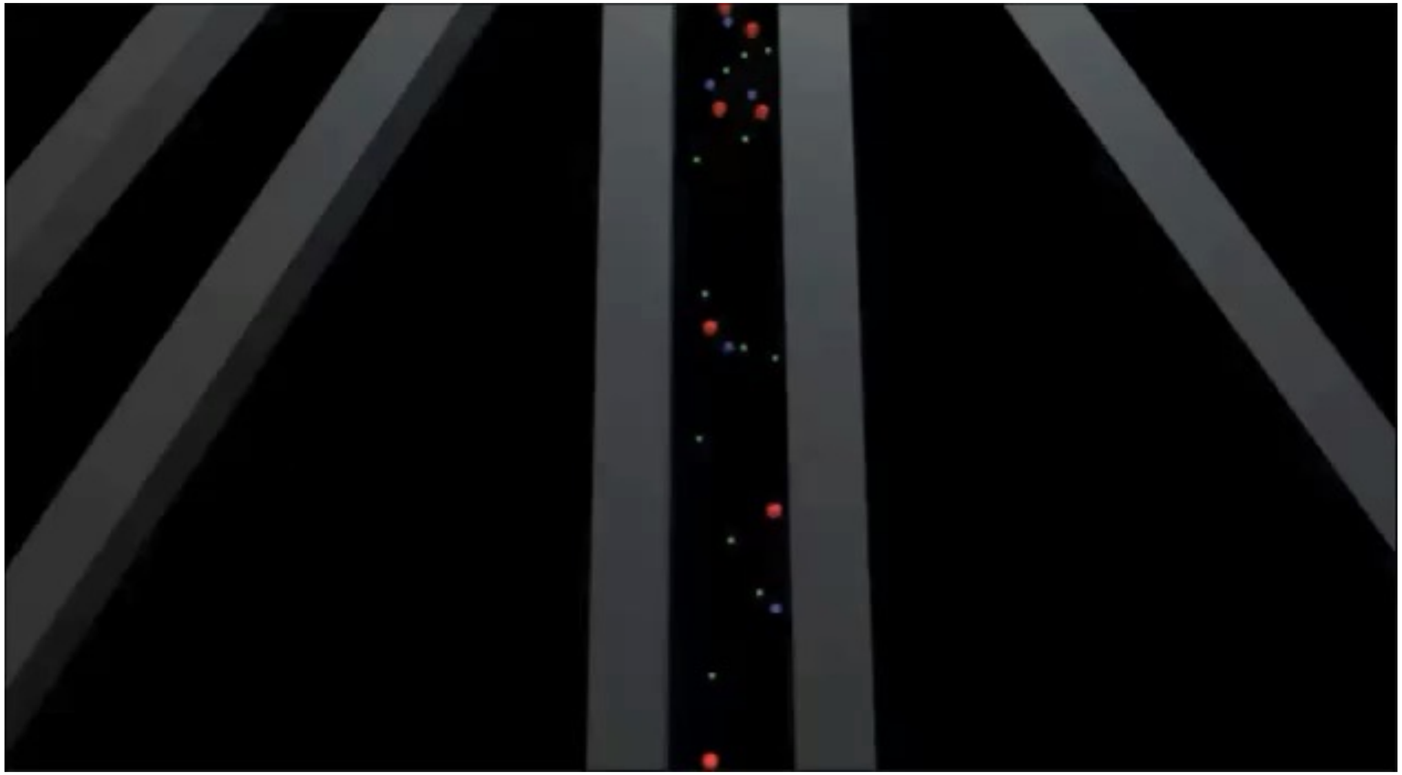


The critical particle radius r where all particle bigger than that are systematically displaced is r=g/3

Huang LR, Cox EC, Austin RH, Sturm JC (2004) *Science* 304:987–989

Particle in laminar flows – Tango array

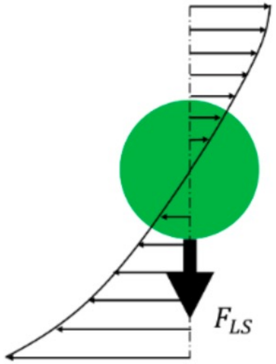




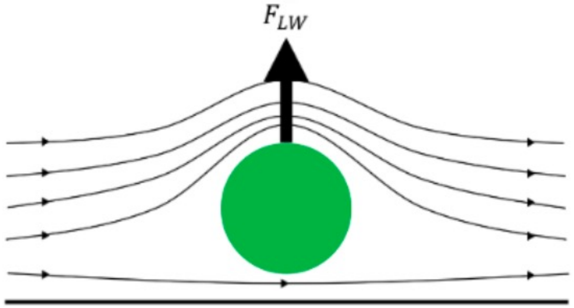
Huang LR, Cox EC, Austin RH, Sturm JC (2004) *Science* 304:987–989

Focusing of particles due to Wall-induced lift force (if flow is fast enough)

When particles are in a fast flow, the shear gradient creates hydrodynamic effects.



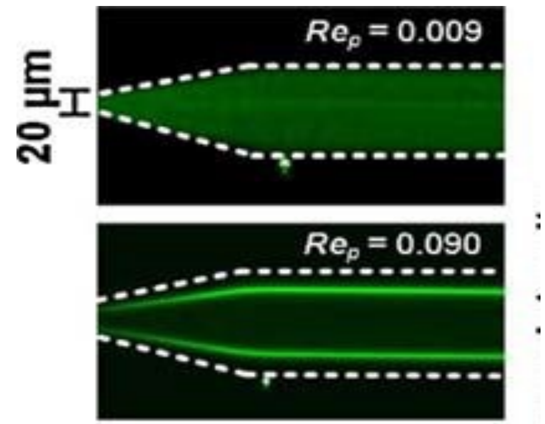
Rotation due to different speeds generates Magnus force pushes towards walls

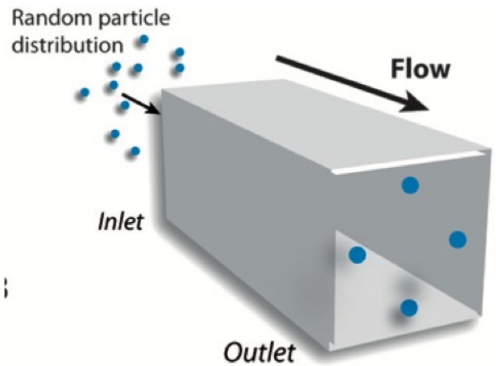


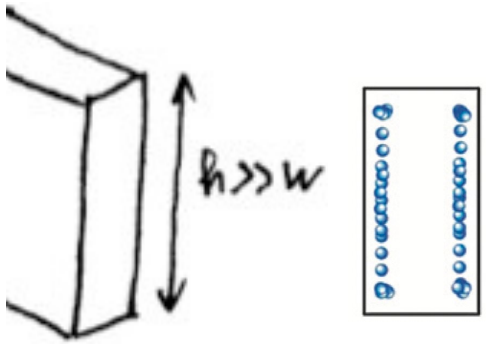
asymmetric wake induced around particles generates a wall-induced lift force

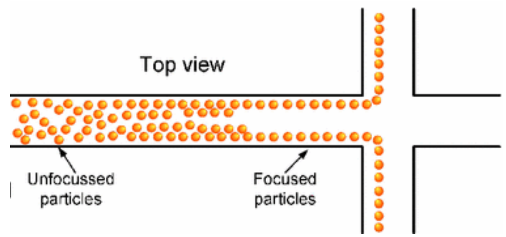
Equilibrium positions when sum of lift forces is zero

$$F_{lift}(z) = \frac{2 \cdot \rho \cdot v_0^2}{h^2} r^4 \bullet f(z) \quad \propto r^4$$







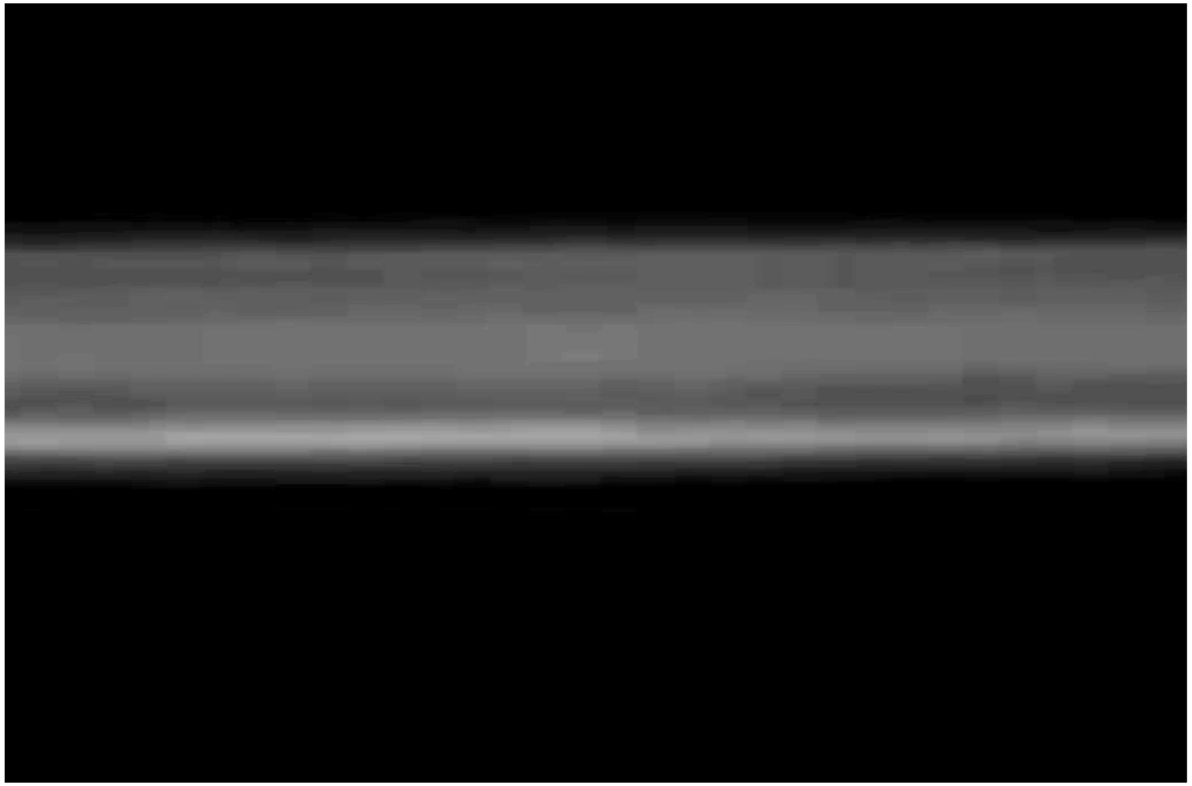


For a rectangular channel, the four equilibrium positions are situated at the center of the walls

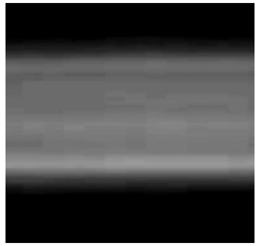
Bhagat et al, Microfluid Nanofluid (2008) 7:217–226

Shear- and wall-induced forces on particles, at high speed when inertia and shear become important

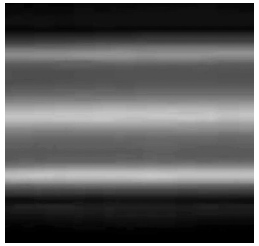
Increasing flow rate (increasing Re)

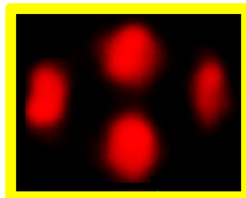


Low flowrate



High flowrate

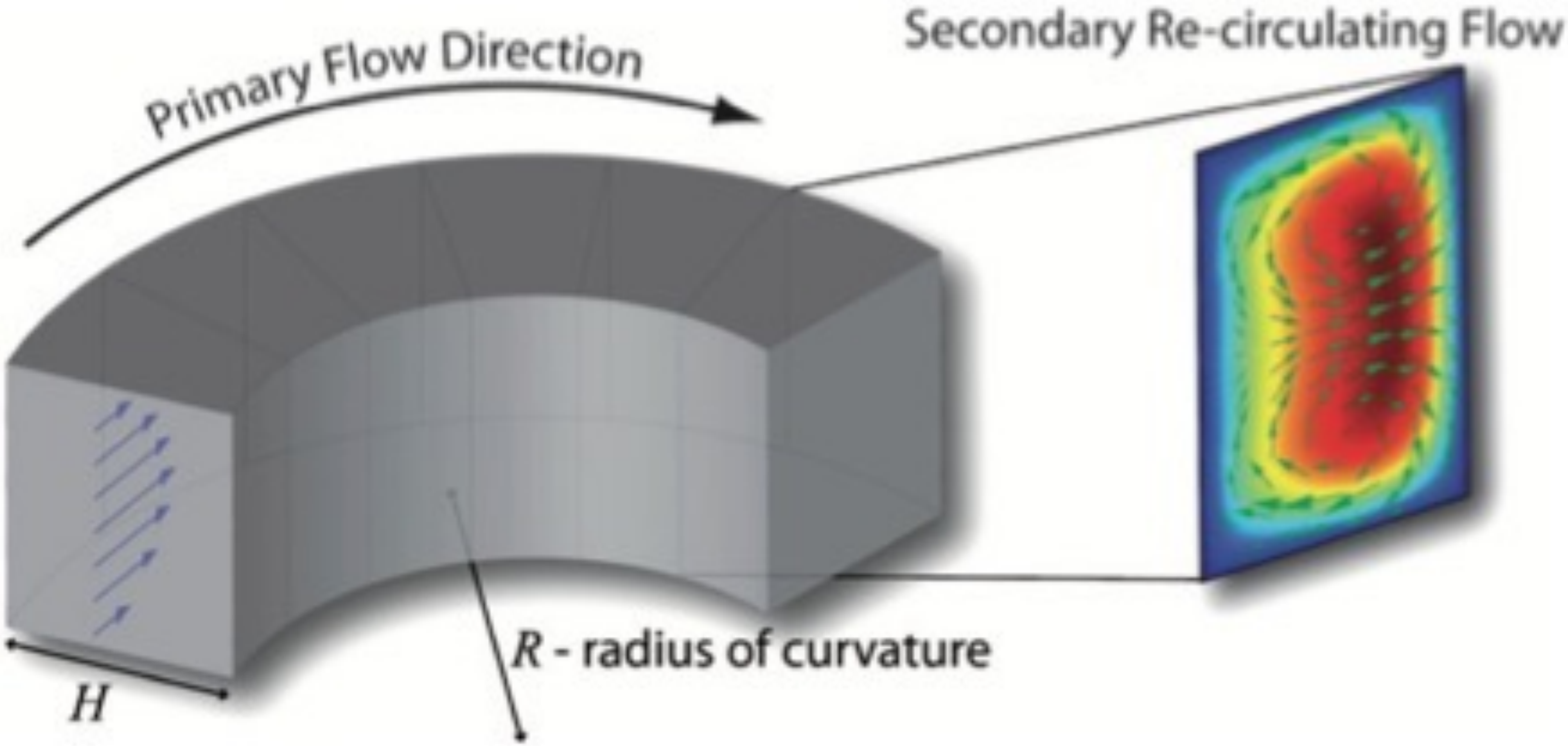




“Inertial microfluidic devices work within an intermediate Reynolds number range (1<Re <100) between Stokes and turbulent regimes.”

J. Zhang, W. Li, G. Alici, “Inertial Microfluidics: Mechanisms and Applications” (2017), pp. 563–593.

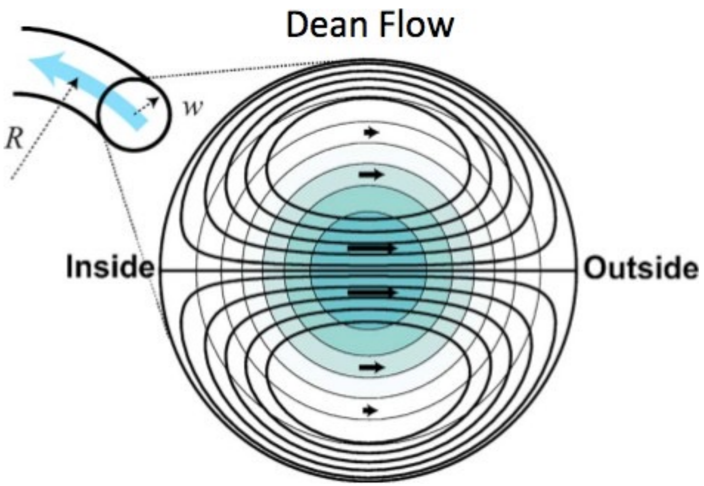
Inertial effects in curved channels: Dean flow



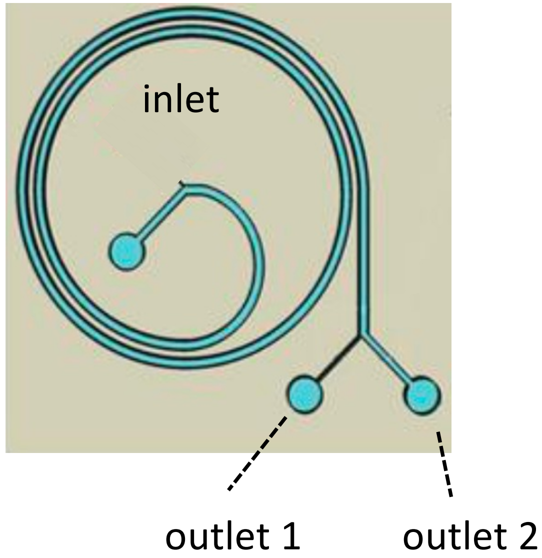
Di Carlo, "Inertial Microfluidics", Lab on a Chip (2009) 3038

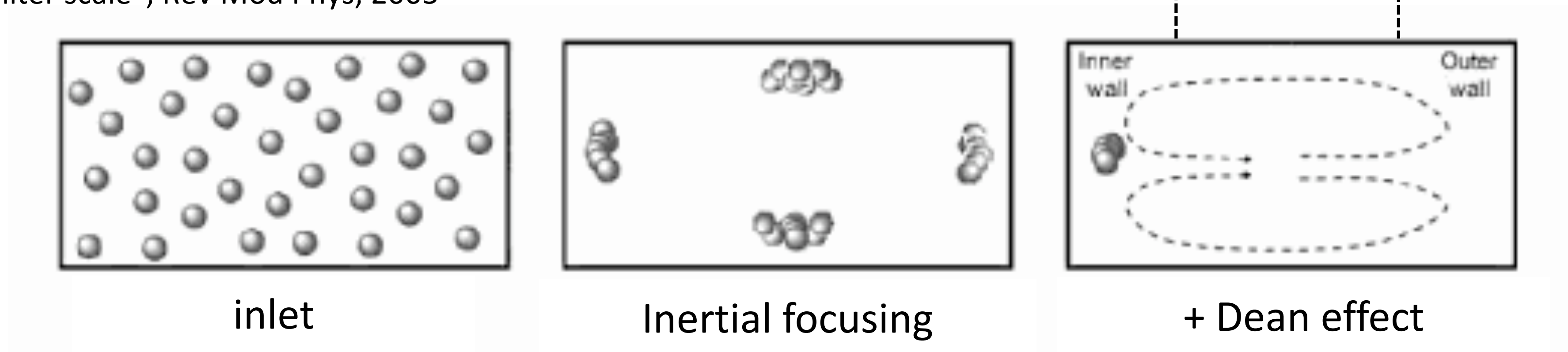
Flow in curved channels: Dean effect

If the channel is running in a curve, a recirculating flow vortex is created. The particles that are in equilibrium positions of the lift force are pushed on the inner side position by the vortex.



T. M. Squires and S. R. Quake: "Microfluidics: Fluid physics at the nanoliter scale", Rev Mod Phys, 2005





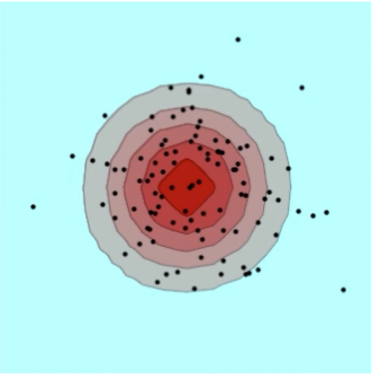
6. Diffusion and Mixing

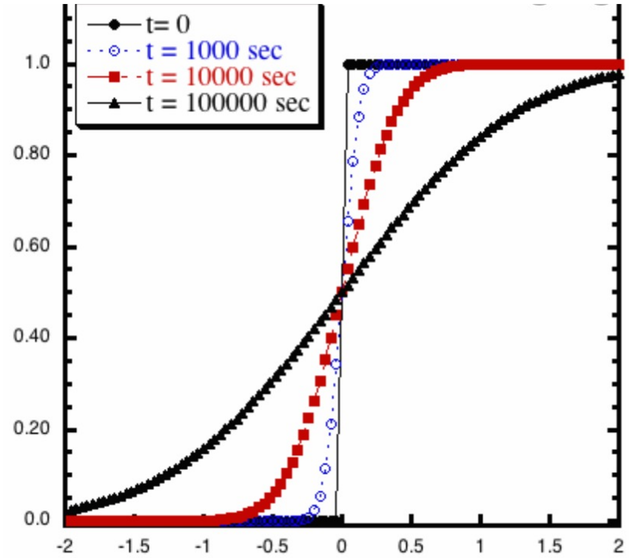
Diffusion of molecules in liquids

Diffusion: Fick's law

$$J = -D_c \frac{\partial C}{\partial x}$$

J: particle flux in #/m²s
C: concentration
D_c: diffusion coefficient in m²/s
η: viscosity

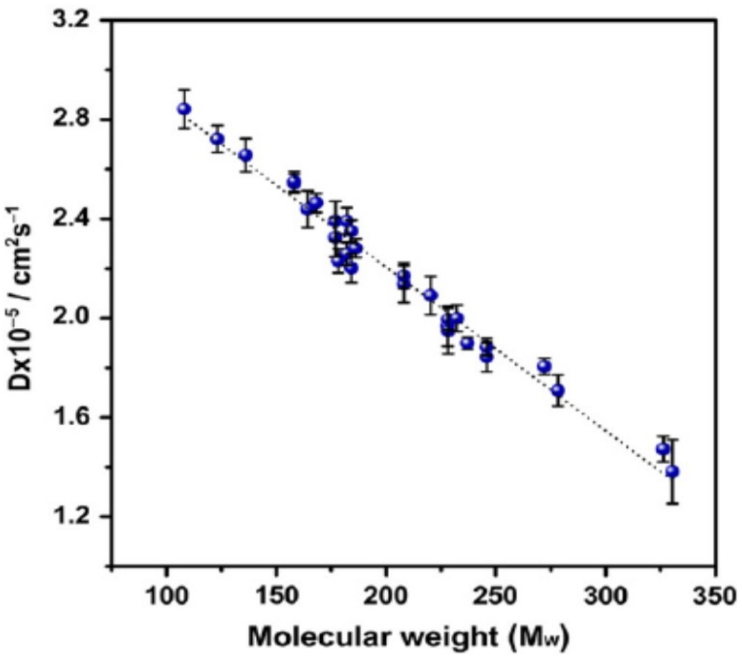




Diffusion coefficient

$$D_c = \frac{k_B T}{6\pi\eta r}$$

$$D_c \propto L^{-1}$$



Diffusion distance and time:

$$x_D = \sqrt{2D_c t}$$

$$t_D = \frac{l^2}{2D_c}$$

$$D_c \propto l_d^2$$

Diffusion of molecules in liquids

Molecule / particle	*Typical size*	*Diffusion coeff. in water*	x_D *in 1s*	t_D *for 50 µm*
Solute ion	0.1 nm	2000 µm^2/s	45 µm	0.6 sec
Small protein	5 nm	40 µm^2/s	9 µm	30 sec
Virus	100 nm	2 µm^2/s	3 µm	10 min
Bacterium	1 µm	0.2 µm^2/s	0.6 µm	104 min
Cell	10 µm	0.02 µm^2/s	0.2 µm	1000 min

Heat diffusivity in water	-	D_{th}=1.4 10^5 µm^2/s	530 microns	0.01 sec

Diffusion layer in microelectrodes for electrochemical sensors (eg glucose, dissolved O_2)

Electrochemical (amperometric) sensors consume molecules in the surroundings. These are then replenished by diffusion. The goal is to measure concentration of a species $C_{species}$, by measuring an electrical current.

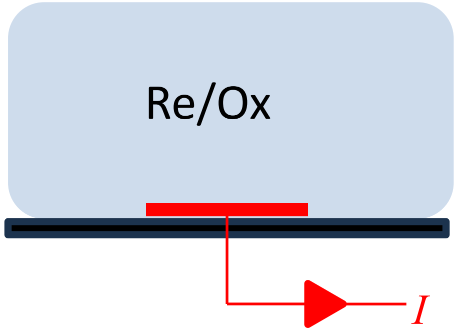
Charge transfer current at the surface of an electrode with redox reaction taking place:

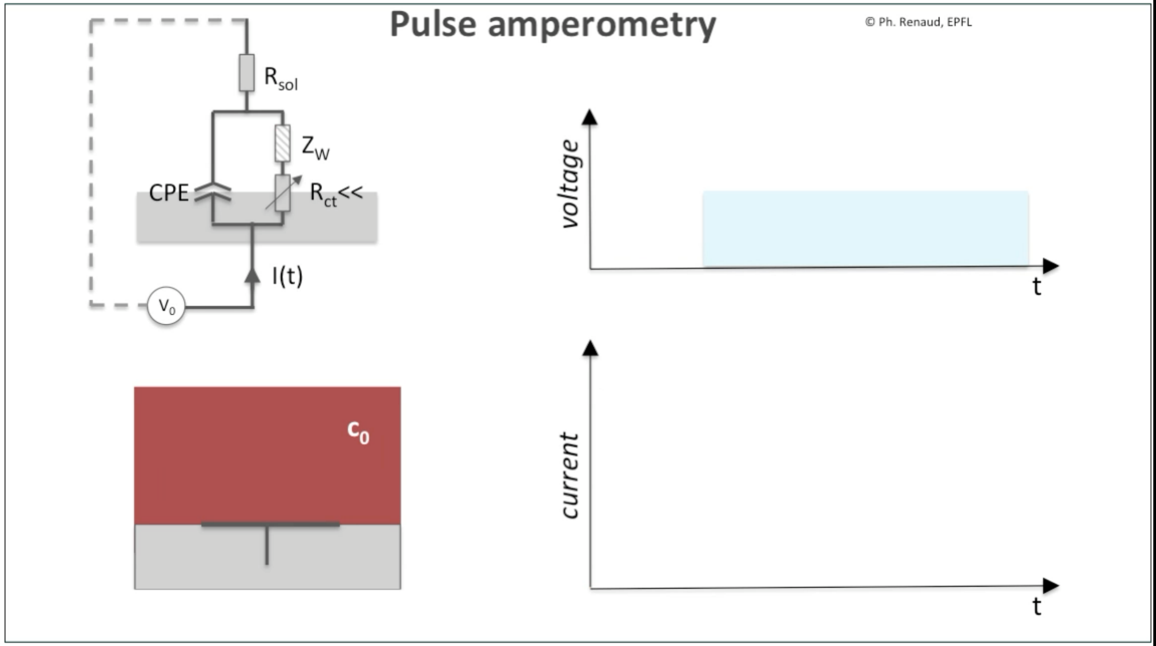
$$I = n \cdot F \cdot A \cdot J_{mol}$$

n: # electron in the reaction
A: area of electrode
F: Faraday constant

The measured electrochemical current is directly related to the molecular flux J_{mol} (that feed the redox reaction)

$$J_{mol} = D_{diff} \frac{\partial C(x,t)}{\partial x}$$





Diffusion layer in microelectrodes

1. For a single **large** circular **electrode**:

$$J_{mol} = D_{diff} \frac{\partial C(x,t)}{\partial x}$$

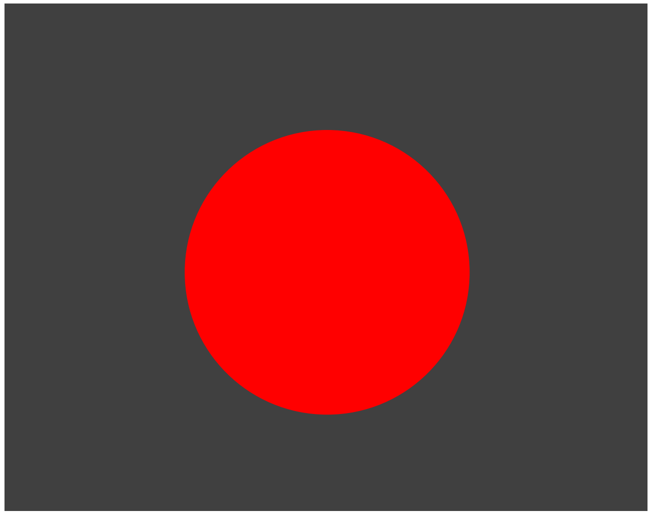
Concentration gradient at electrode surface: $\left.\frac{dC}{dx}\right|_{x=0} = \frac{C_0}{\delta}$

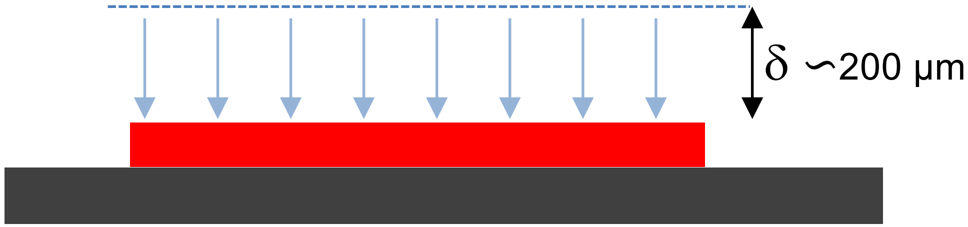
δ is the thickness of the **diffusion layer** (about 0.2 mm)

$$I_{\lim} = n \cdot F \cdot \pi r^2 \cdot D_{diff} \frac{C_\infty}{\delta}$$

$$\frac{I}{A} \propto \frac{1}{\delta}$$

$$I \propto A \frac{1}{\delta}$$





Diffusion layer in microelectrodes

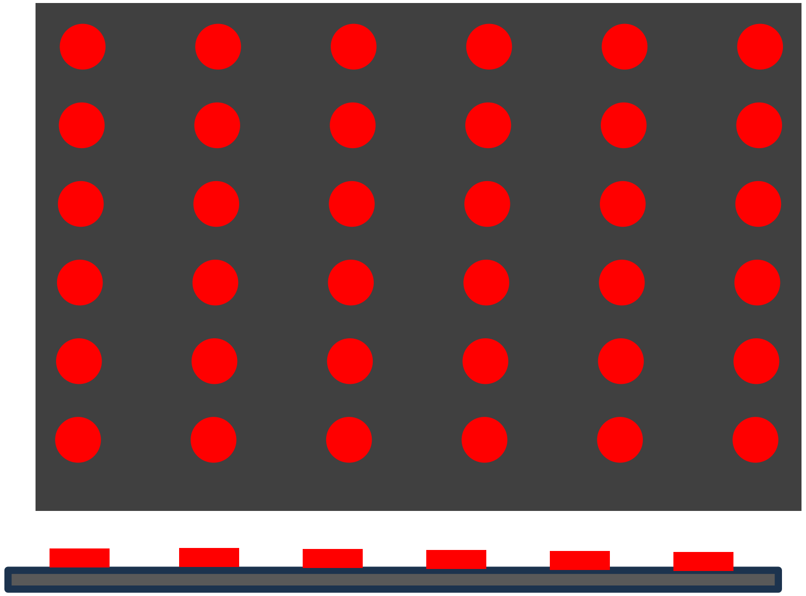
$$J_{mol} = D_{diff}\frac{\partial C(x,t)}{\partial x}$$

2. For array of circular **micro-electrode**
(radius r << diffusion layer δ)

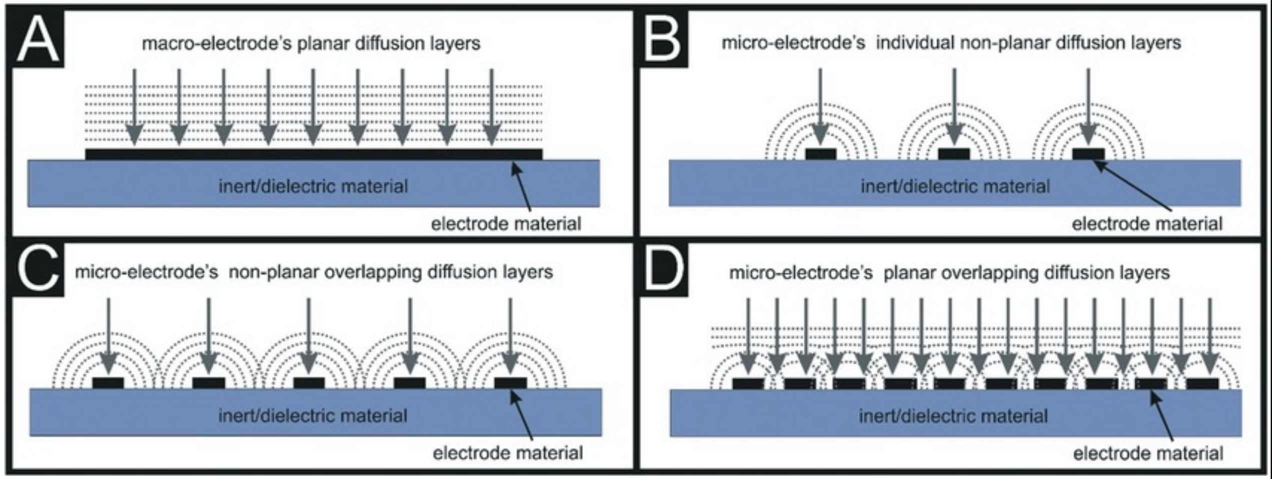
$$I_{\lim} = n \cdot F \cdot \pi r^2 \cdot D_{diff}\frac{C_0}{r}$$

$$\frac{I}{A} \propto \frac{1}{r}$$

$$I \propto A\frac{1}{r}$$



$r << d$, so Current is much larger with many small electrodes of total area A than with a single large electrode of area A!

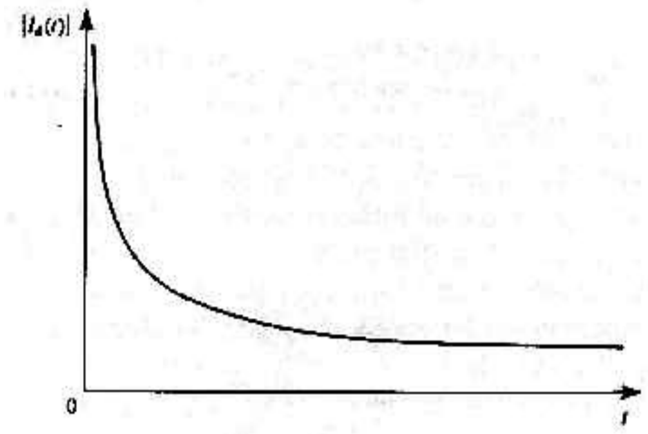


https://doi.org/10.1039/D0EW00407C

Response time of microelectrodes

The current around a macroscopic electrode is given by the Cottrell equation:

$$I(t) = nFAD_c C_0 \left(\frac{1}{(\pi D_c \cdot t)^{1/2}} + \frac{1}{\delta} \right)$$



Time evolution of electrochemical current

<u>There are two regimes</u>:

- Initially, when V is turned on, the current is very high until a **steady diffusion layer** is formed
- Then the current is diffusion limited (better for steady measurements)

With micro-electrodes, the diffusion layer thickness is limited to the **radius of the electrode**. Thus the **time** to reach the limiting current is much faster than for macro electrodes (and the **current density** is higher!)

The related time to reach this layer is given by

$$x_{,\max} = \sqrt{2D_{diff} \cdot \tau} \quad \text{then} \quad \tau \approx \frac{r^2}{2D_{diff}} \qquad \tau \propto r^2$$

<u>Numerical</u>: $D_{diff} = 10^{-9}$ m^2/s

$r_0 = 10$ µm $\tau = 50$ ms
$r_0 = 5$ µm $\tau = 12.5$ ms
$r_0 = 1$ µm $\tau = 0.5$ ms

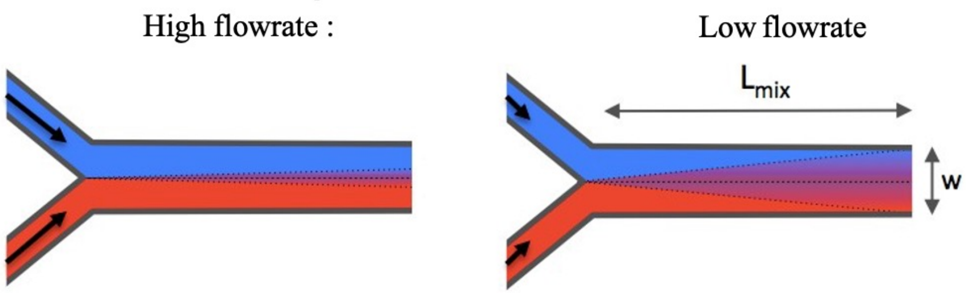
How to Mix fluids in Laminar Flow conditions?

At low Reynolds numbers, mixing occurs only by *diffusion*, not by convection (turbulence)

So: need to make *flows thinner* so that ***diffusion*** *times are shorter*

Good overview:
https://ocw.mit.edu/courses/2-674-micro-nano-engineering-laboratory-spring-2016/f28f8672e6386d65276d2e6776ae8611_MIT2_674S16_MicrofluidcMix.pdf

Mixing in laminar flow conditions



Time for stirring (mass transport): $t = \frac{l}{v}$

At low Reynolds numbers, mixing occurs only by diffusion, not by convection (turbulence)

Time to diffuse across the channel width:

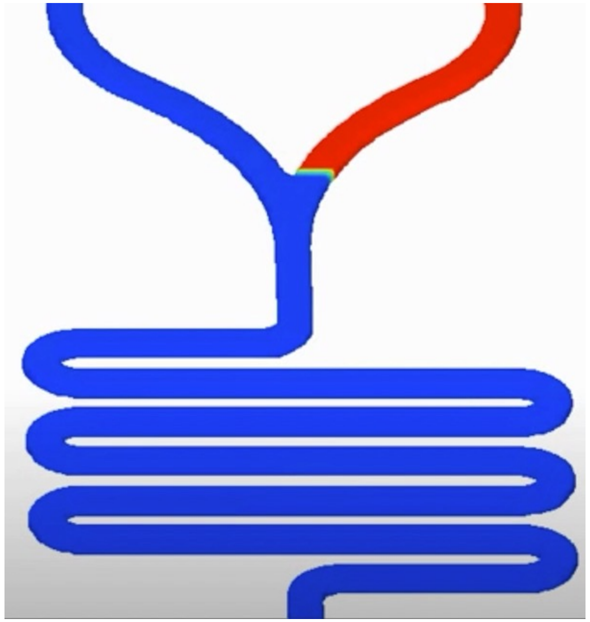
$$\tau_D \sim \frac{w^2}{D}$$

In this time, the fluid moves a distance:

$$Z \sim \frac{v\, w^2}{D}$$

$$\frac{Z}{w} \sim \frac{v\, w}{D} \equiv Pe$$

w channel width
D diffusivity
U_0 velocity



Peclet Number: $\text{Pe} = \frac{l\, v}{2D_C}$

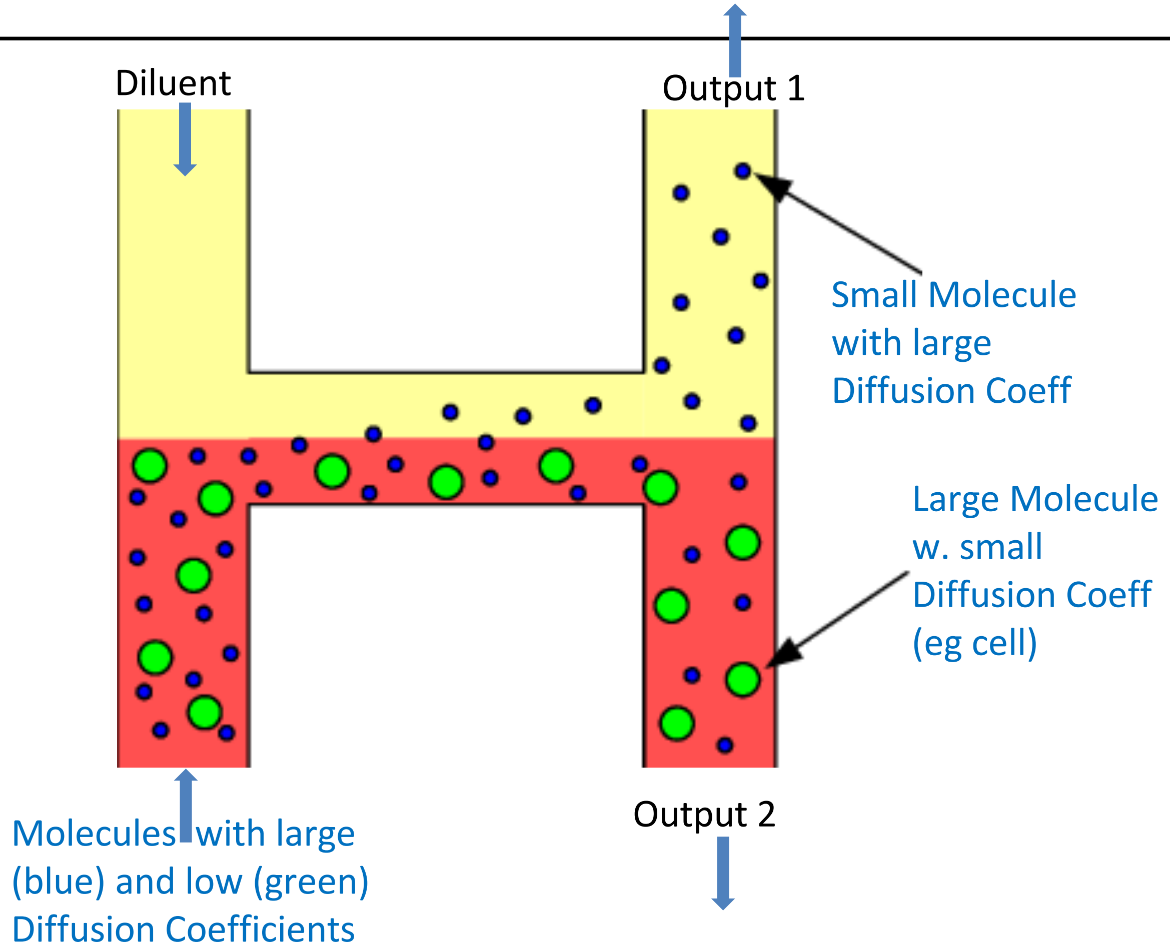
Ratio : convection trsp. / diffusion trsp.

High Peclet #: dominated by convection.
Low Peclet #: dominated by diffusion

The H-filter: exploiting diffusion and laminar flow

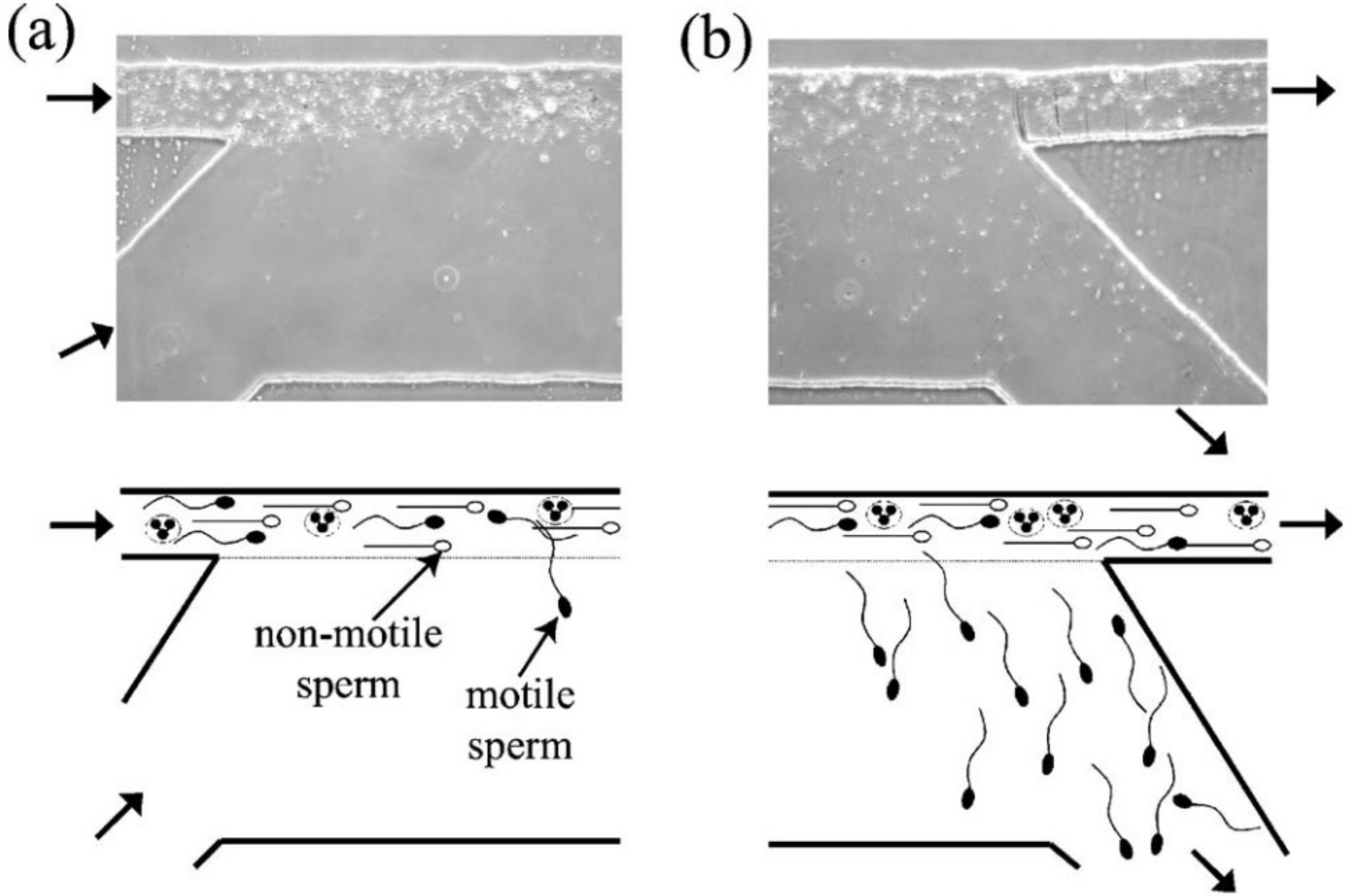
<u>Principle</u>: filtration without membrane

- Two laminar flows are put in contact
- High diffusion coefficient molecules/ particles will diffuse laterally
- Low diffusion coefficient partickes mostly remain in initial flow
- The two flows are then separated



Contact length	small molecule (D_c=1000 μm²/s)	20 μm cell ($D_c=10^{-2}$ μm²/s)
l = 10 μm	t_d = 0.01s	t_d = 1 min
l = 100 μm	t_d = 1s	t_d = 100 min

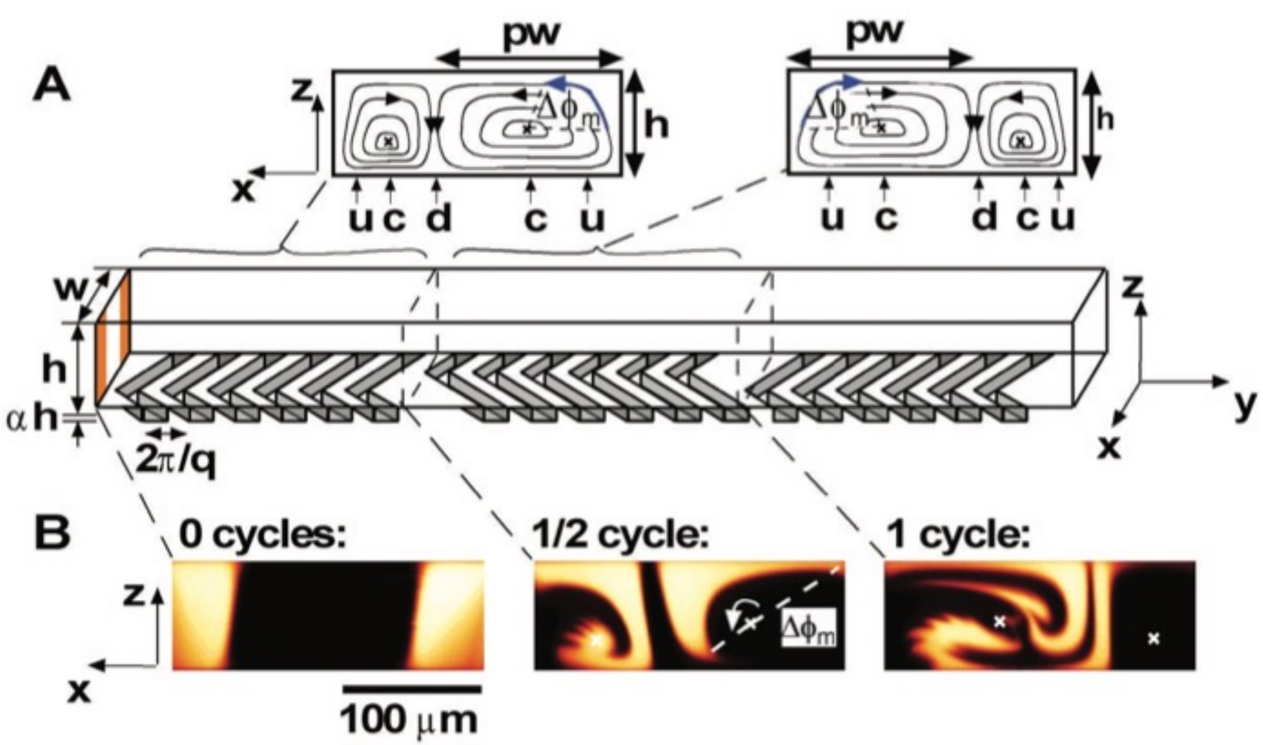
- «Applying microfludic chemical analytical systems to imperfect samples», P. Yager et al., MicroTAS 98
- Brody et al., 1996; Brody and Yager, 1997
- "Cell sorting in microfluidic systems", P. Tellemar et al, MicroTAS 98
- Commercialized by Micronics

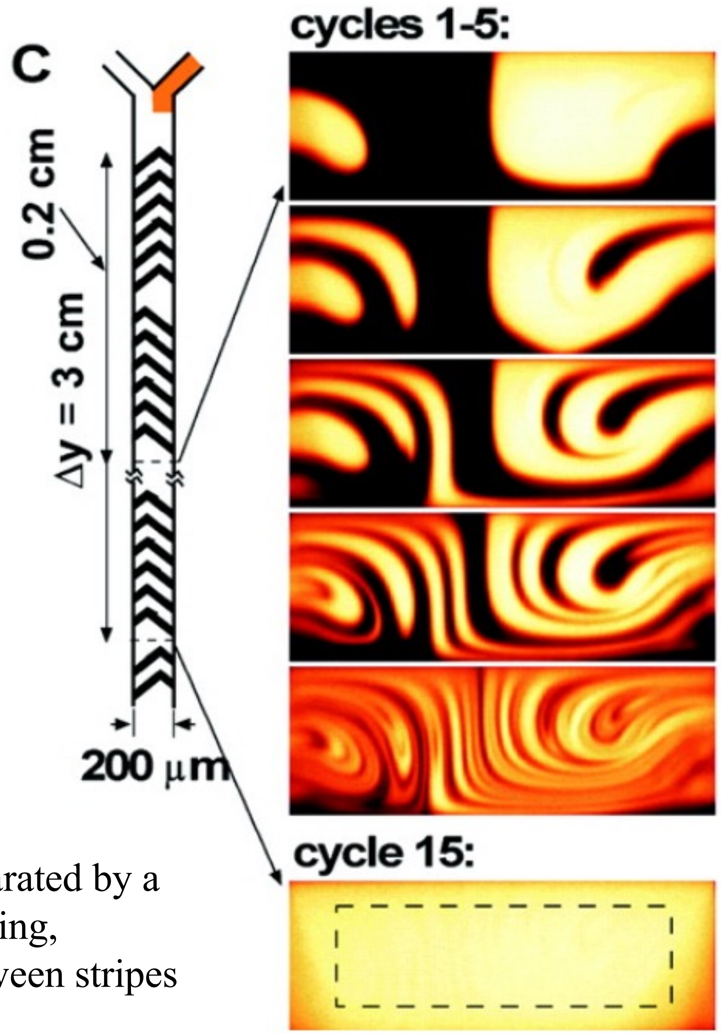


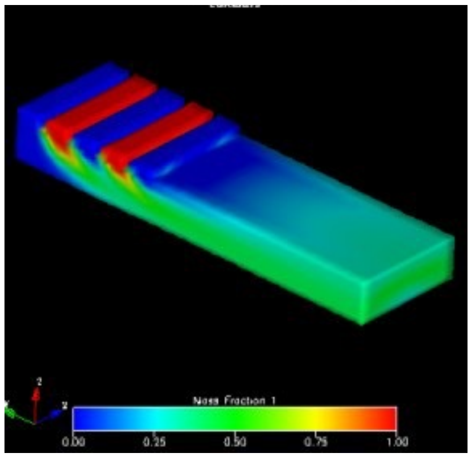
"While the traditional H filter relies upon particle diffusivity differences for differentiation, this device exploits the fact that motile sperm disperse across and homogenize the channel much more rapidly than nonmotile ones, which spread via diffusion alone."

B. Co, Schuster *et al.*, 2003
Anal. Chem.2003,75,1671-1675

Mixing by chaotic advection







After N cycles requiring $_{cyc}NL_{cyc}/U$, where L_{cyc} is the cycle length, stripes are separated by a distance $h_{eff}h/2_N$. Following the above reasoning, mixing occurs when the time to diffuse between stripes h_2/D is comparable to the cycle time ,

$$N_{chaotic} = \ln \mathrm{Pe}$$

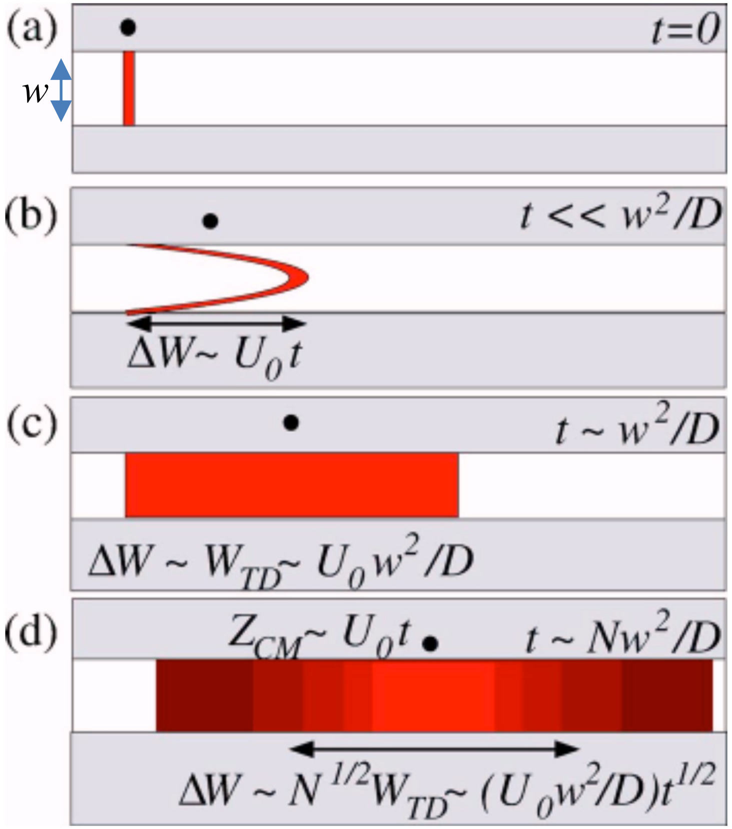
Stroock, Science 2002

Induce recirculation between sections, so then diffusion can be more effective

Microfluidic mixing

Taylor dispersion (acts in the direction of flow)

The role of convection in dispersing inhomogeneous flows



- Thin stripe of tracer spans circular channel of radius w
- Pressure driven flow stretches it into a parabola
- Molecular diffusion then smears the parabolic stripe into a plug of width $W_{TD} \sim \frac{U_0 w^2}{D}$ after a time τ_D
- Each stripe within the plug goes through the same process: convectively streched and diffusively smeared.
- After N steps, the initally thin stripe evolves to a Gaussian with width:

$$\sqrt{< W^2 >} \sim \sqrt{N} \quad W_{TD} \sim \sqrt{\frac{U_0^2 w^2}{D} t}$$

Effective **long-time** axial diffusivity

$$D_z \sim \frac{U_0^2 w^2}{D} \sim Pe^2 D$$

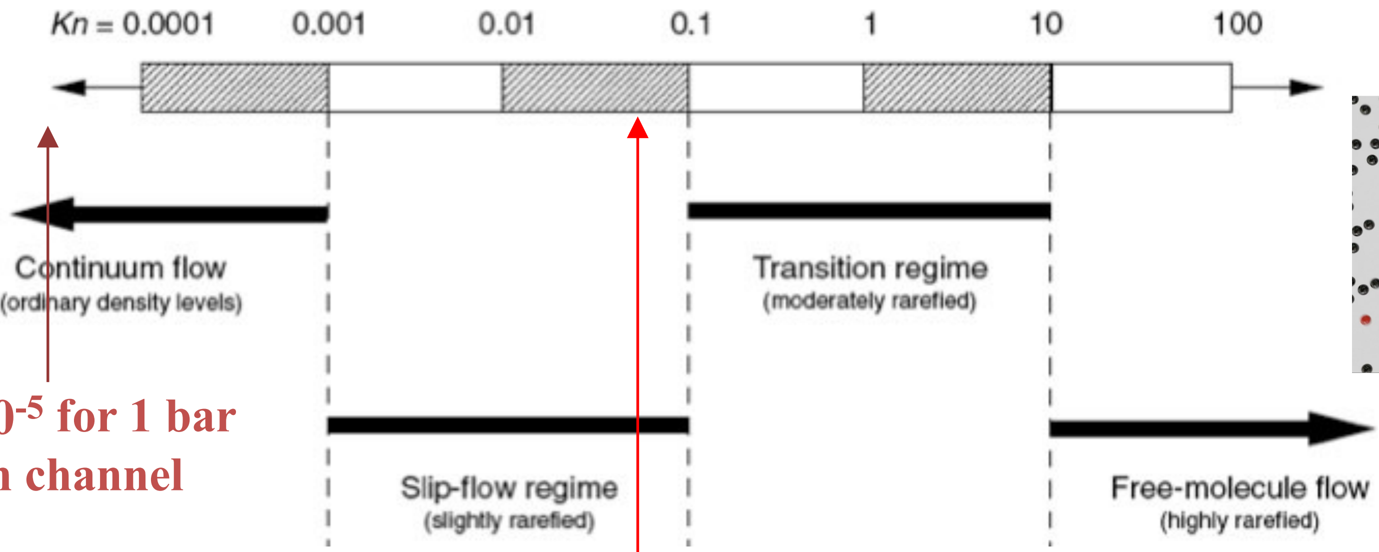
T. M. Squires and S. R. Quake: Microfluidics:
Fluid physics at the nanoliter scale, Rev Mod Phys, 2005

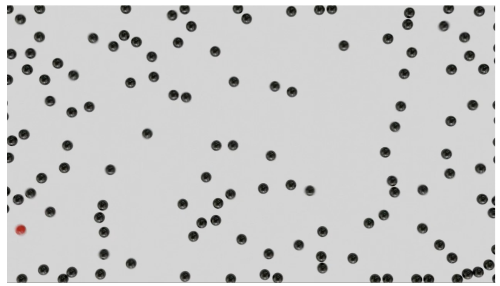
7. Gas flow – Knudsen number
Transitions from fluid to discrete molecules

Gas flow - Knudsen number – transition from fluid to discrete molecules

Mean free path of gas molecules : $\lambda = \frac{\eta}{\rho}\sqrt{\frac{\pi}{2R_0T}}$ air, 20°C, 1 atm: $\lambda = 66\ nm$

The Knudsen number is defined as : $K_n = \frac{\lambda}{d}$ d : hydraulic diameter, or height of slit





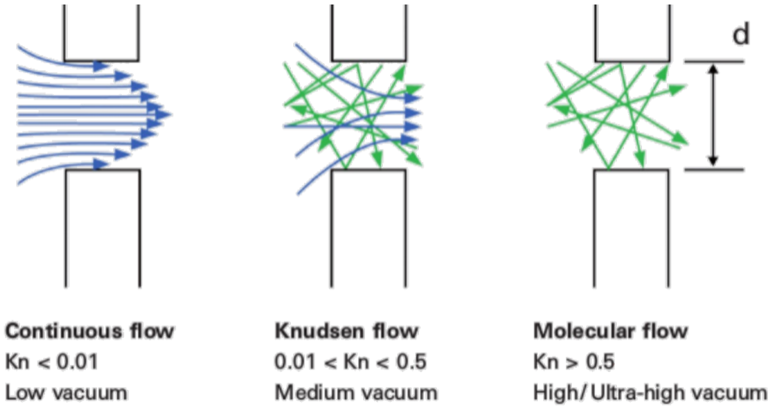
K_n = 7.10^-5 for 1 bar and 1 cm channel

K_n = 0.07 for 1 bar and 1 micrometer channel

- At low K_n, the gas can be considered as a fluid => normal fluid dynamics (Navier-Stokes) and gas theory
- At high K_n numbers, the gas can no longer be considered as a continuum

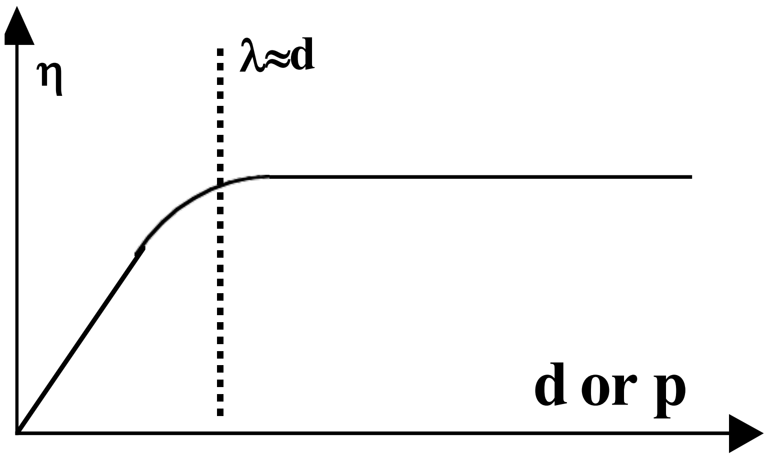
Effective viscosity in molecular regime

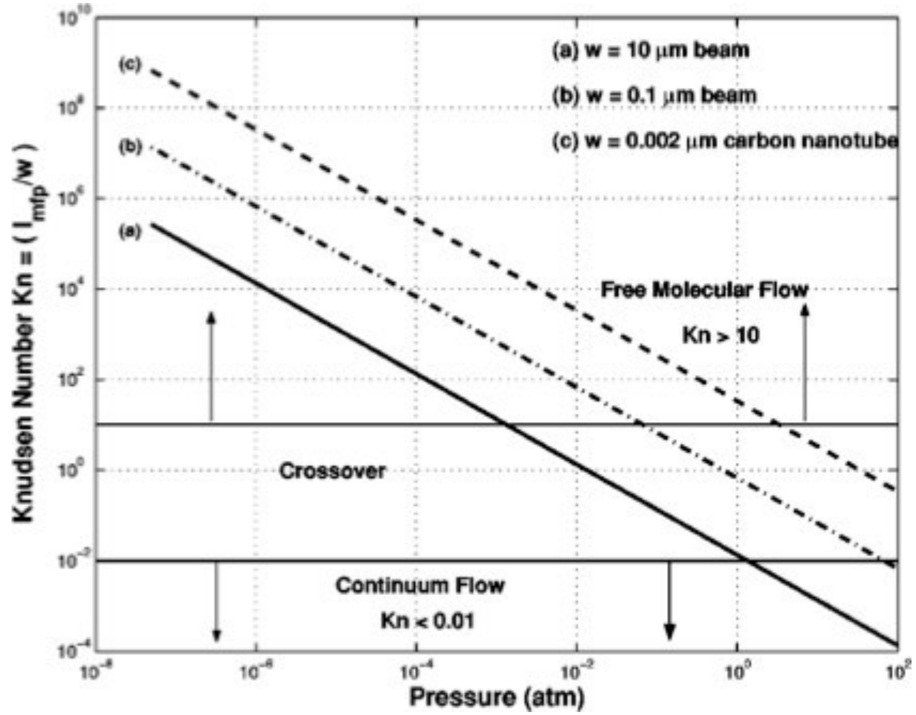
When entering the molecular regime, the gas viscosity drops because of sparse distribution of molecules in the volume of interest. The effective viscosity η_{eff} deviates from bulk viscosity η :



$$\eta_{eff} = \frac{\eta}{1 + 9.6 K_n^{1.2}}$$

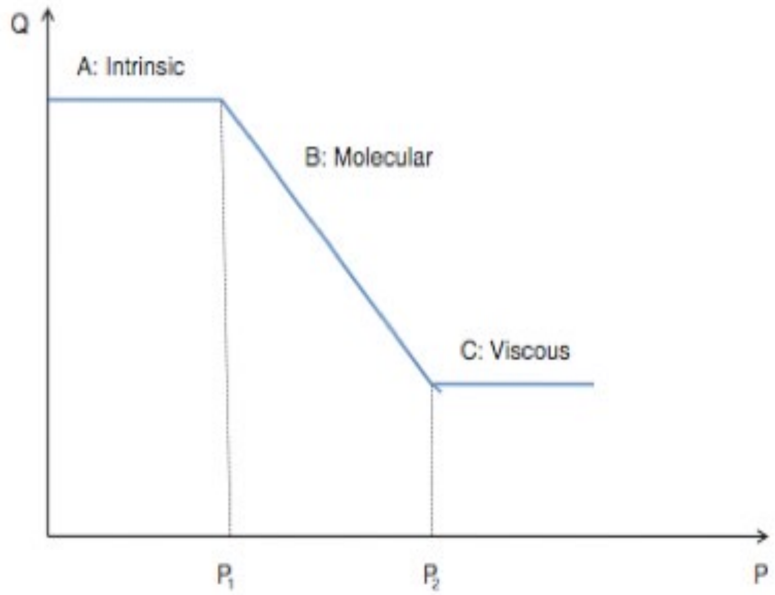
- Down to about 10 μm (objects or channels) the continuum approximation applies at atmospheric pressure





- A Knudsen number can be associated to the air surrounding an object. For a beam of width *w* moving in the vertical direction, $K_n = \lambda / w$
- A **nanosized beam is thus in the molecular regime** (*Bhiladvala, Phys Rev. E 69, 036307, 2004*)

Gas damping – quality factor Q as a function of pressure



Quality factor vs pressure for a resonator in air

Regime A: the pressure is very low, below 10 Pa. The quality factor is limited by intrinsic mechanical losses (anchor, thermo-elastic, etc)

Regime B: the pressure is in the range from 10 to 1000 Pa. In this regime, air damping is dominant and the quality factor is very sensitive to pressure. Air molecules are so apart from each other that they do not interact. Air damping occurs due to momentum transfer during collision of the molecules with the moving structure.

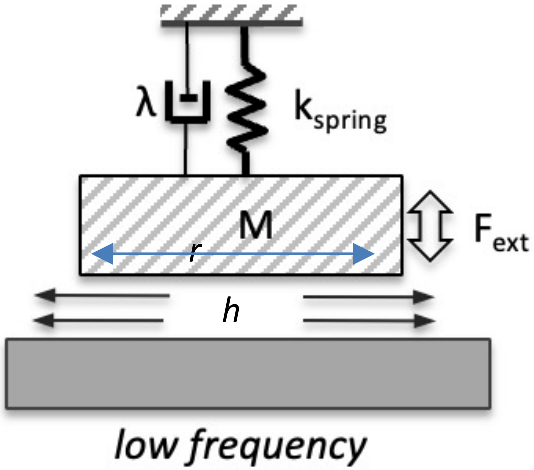
Regime C: the pressure is higher, approximately above a 1000 Pa. In this case, air molecules interact with each other and the air acts as a viscous fluid.

In the molecular regime (B), the quality factor Q scales like 1/P

In the viscous regime, the Q factor is related to viscosity of gas and geometry of structure

Squeeze film damping in MEMS (e.g. capacitive microphone, accelerometer)

Damping coefficient for two parallel disks of area A and spacing h, need to move air out

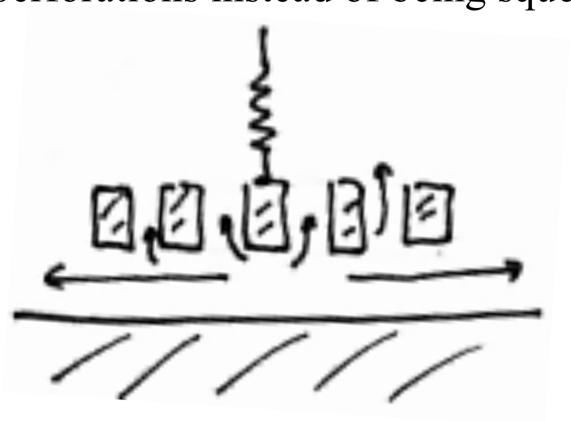


$$\lambda_{film} = \eta \frac{3\pi r^4}{2h^3} \quad [Ns/m]$$

$$\lambda \propto \frac{r^4}{h^3}$$

(if move slowly enough for gas to leave/enter …)

If one disk is perforated, the damping can be considerably reduced, as the fluid can escape through the perforations instead of being squeezed out of the disk edges.



exact formulation can be found in Veijola papers

"Model for gas damping in silicon accelerometer", T. Veijola et al., Transducers 97 (1997)

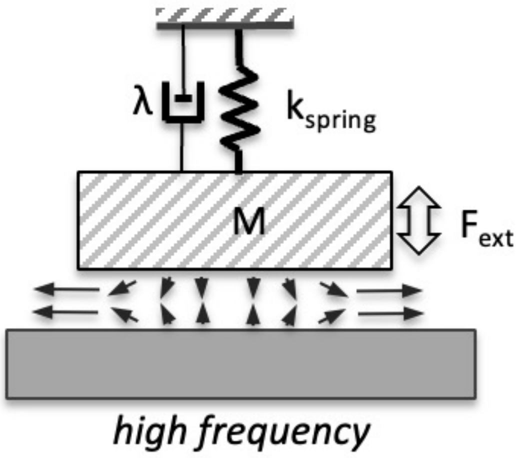
Squeeze film damping at high frequency – "piston" effect

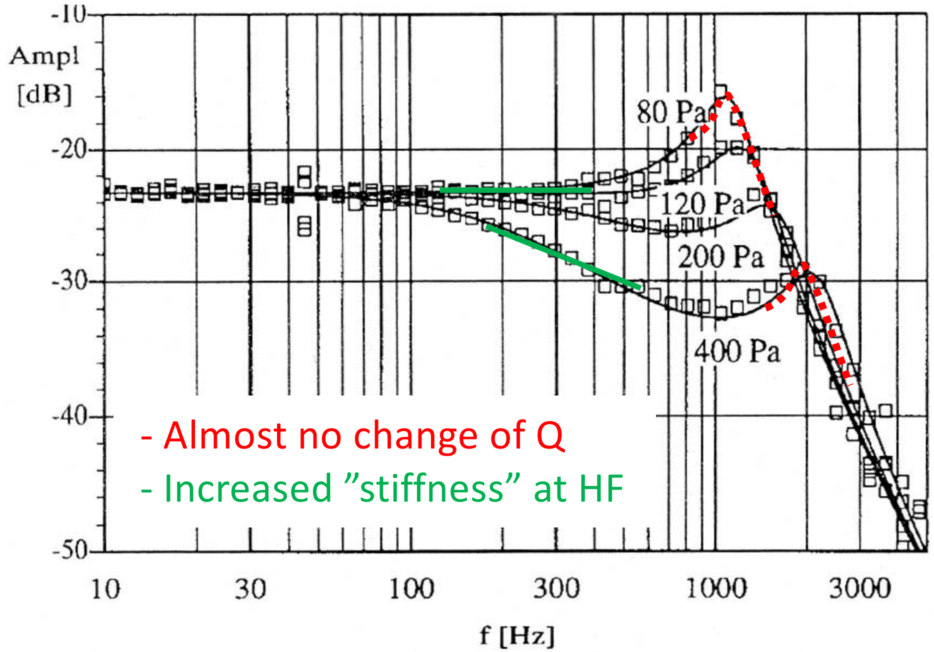
At high oscillation frequencies, the gas does not have time to flow out of the gap. Gas film then acts from the mechanical point of view as a spring. So no damping anymore, but increased stiffness (so lower amplitude)

At high frequency limit, the additional spring force is

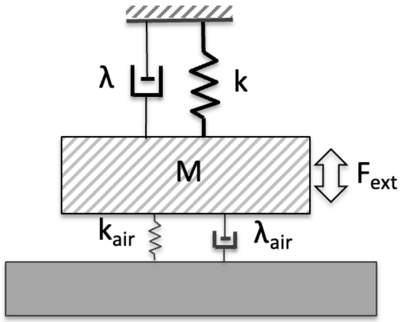
$$k_{air} = \frac{p_0 A}{d}$$

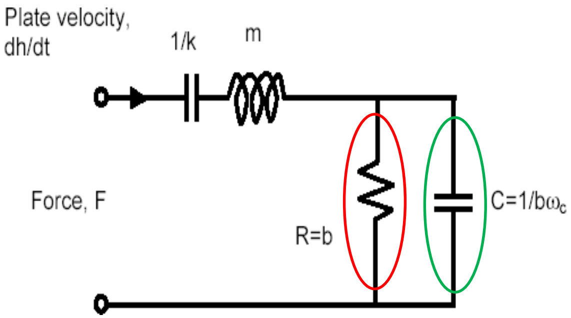
k_{air} depends on frequency!!





Squeeze film damping effect on the frequency response curve of an accelerometer in function of residual pressure



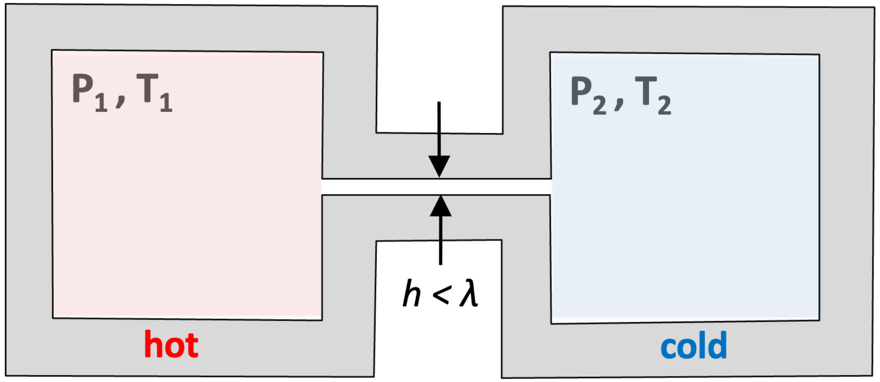


Knudsen pump

When two chambers, at different temperatures, are separated by an aperture of dimension smaller than the mean free path of molecules, there is a pressure gradient at equilibrium.

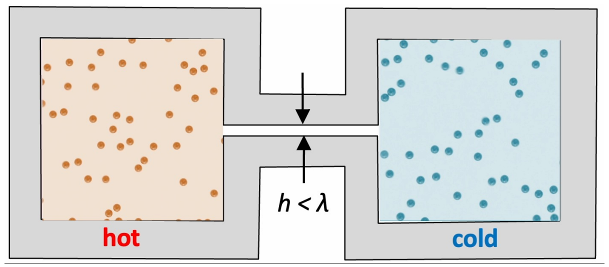
This is known as "thermal transpiration" effect

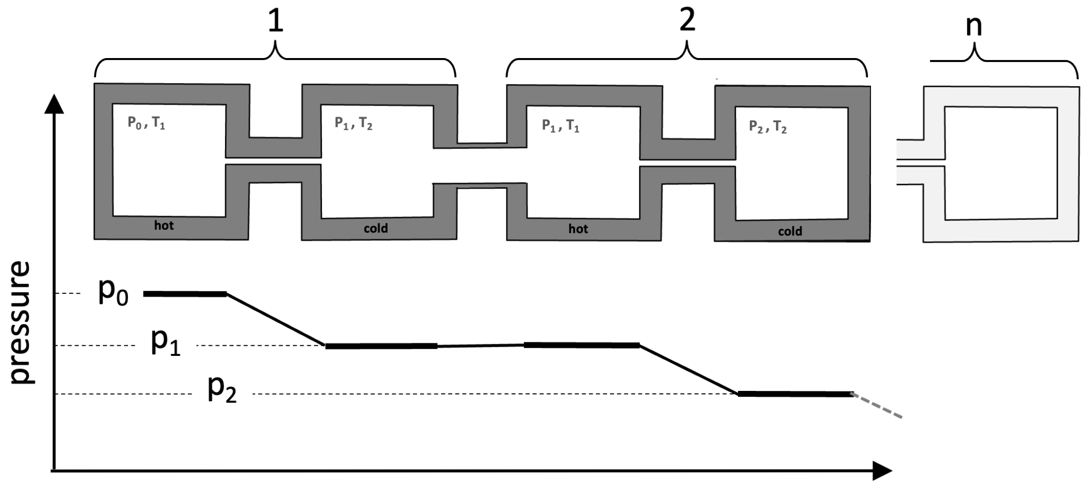
$$\frac{p_1}{p_2} = \sqrt{\frac{T_1}{T_2}}$$



$T_1=400$, $T_2=300$ => $p_2=0.87\ p_1$

By stacking hot and cold cavities linked by narrow and wide channels, it is possible to make a pump.





- *Hobson and Slazman, "Review of pumping by thermal molecular pressure", JVST A 18 (2000) 1758.*
- *N. Gupta et al., J Micromech. Microeng. 22 (2012) p. 105026*

Knudsen pump with 162 stages

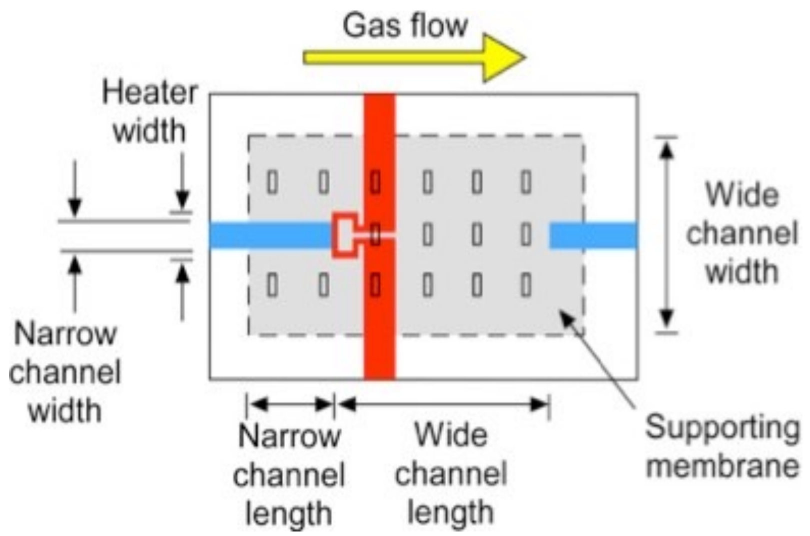
Theoretical pressure ratio for N stages: $$\frac{P_1}{P_2}=\left(\frac{T_1}{T_2}\right)^{\frac{N}{2}}$$

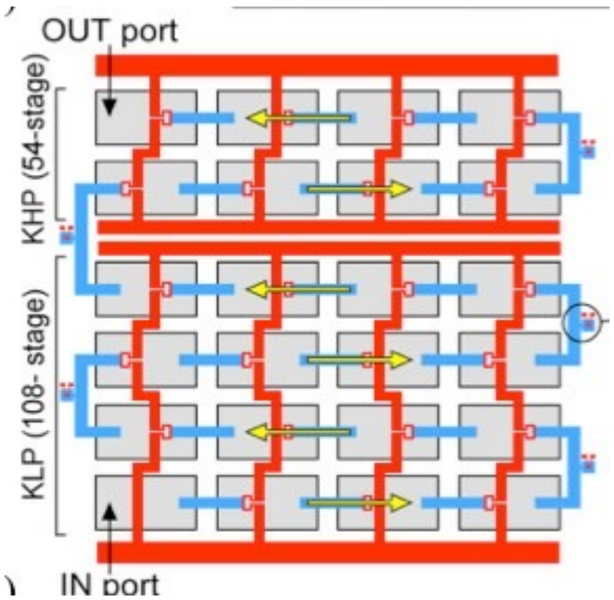
T_1=350, T_2=300, N=100 => p_2= 10^{-4} p_1

- The narrow channels must have a hydraulic diameter less than 1/10 of the mean free path of the gas and the wide channels must have a hydraulic diameter greater than 20 times the mean free path of the gas.

- The maximum operating pressure is defined by the diameter of the narrow channels
- The lowest attainable pressure (best vacuum) is defined by the diameter of the wide channels.

162 stages are cascaded: 54 stages designed for the pressure range from 760 to ≈50 Torr, 108 stages designed for lower pressures (longer mean free path, so larger channels)

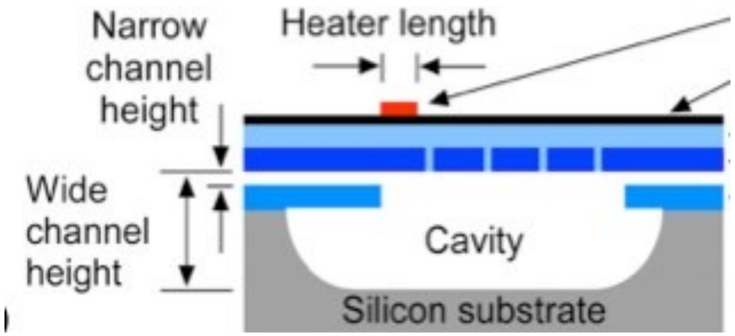


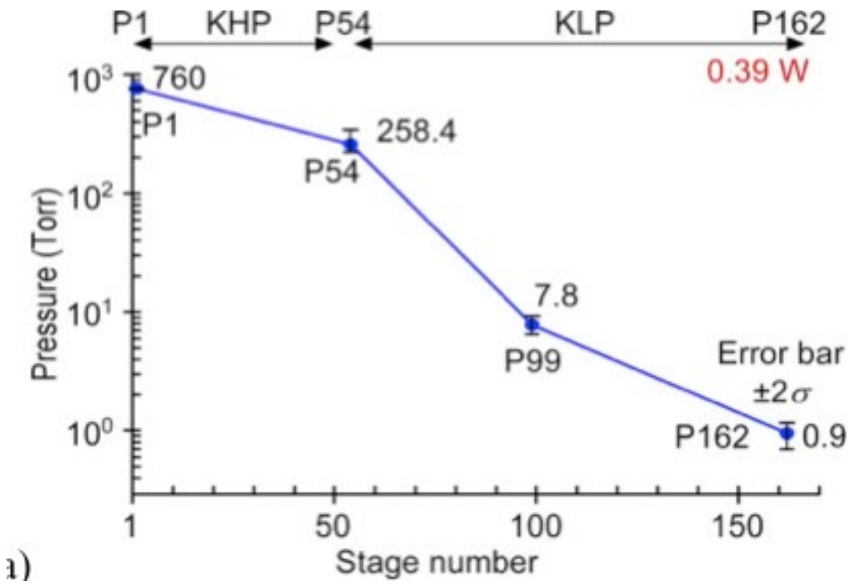


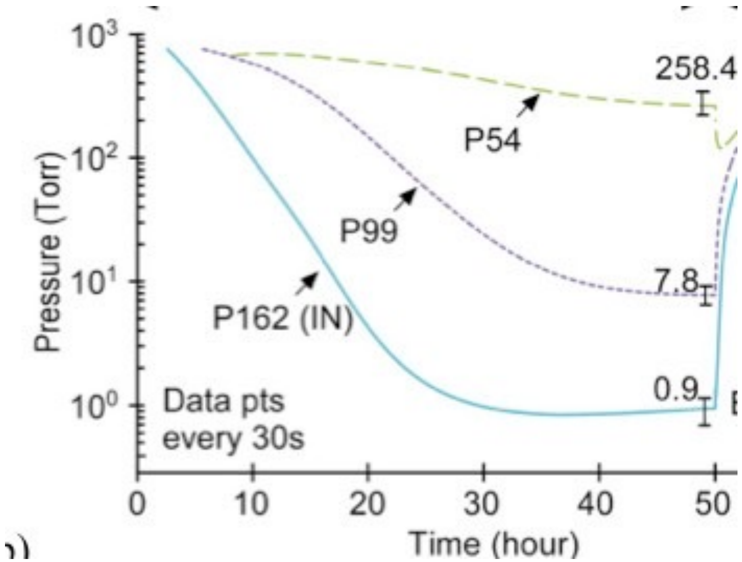
S. An et al., JMEMS, 23,2, 2014 p. 406

Knudsen pump with 162 stages

Part	Narrow channel height (µm)	Wide channel height (µm)	Number of stages	Pressure (Torr)	λ (µm) at 300 K
KHP (760Torr -50Torr)	0.1	30	54	760	0.07
				200	0.28
				50	1.12
KLP (≤50Torr)	1.0	100	108	1	49
				0.1	511



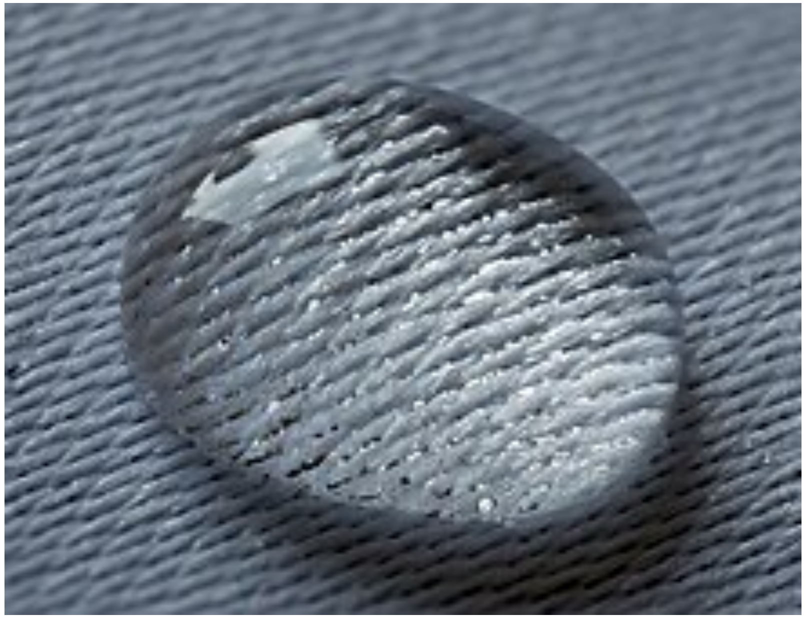




S. An et al., JMEMS, 23,2, 2014 p. 406

Review on Knudsen pumps: Wang et al. Microsystems & Nanoengineering (2020)6:26
https://doi.org/10.1038/s41378-020-0135-5

8. Surface tension and capillary pressure



https://en.wikipedia.org/wiki/Surface_tension

Surface tension

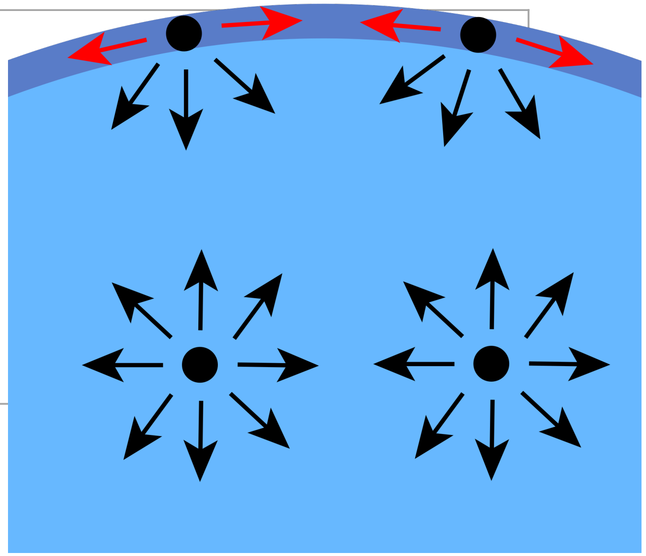
The surface tension at the surface of a liquid and a gas (or at the interface between immiscible liquids) originates from the difference of internal adhesion energy of the liquids and adhesion energy at the interfaces.

It a measure of the excess energy (unrealized bonding energy) present at the surface of a material, compared to the bulk.

Surface tension definition : $\gamma_{lg} = dE_{pot} / dS$

Water in air $\gamma_{lg}=73 \cdot 10^{-3}$ N/m

Ethanol in air $\gamma_{lg}=22 \cdot 10^{-3}$ N/m



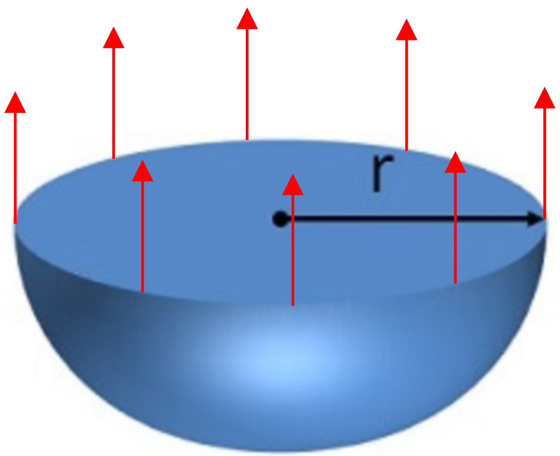
https://en.wikipedia.org/wiki/Surface_tension

<u>Remark</u>: ordinary tap water has only a surface tension of about $40 \cdot 10^{-3}$ N/m, because of impurities

What is the pressure p inside a droplet in air (or inside an air bubble in water) ?

$F_{tot} = \gamma\, 2\pi\, r$

$$p = \frac{F_{tot}}{Area} = \frac{2\,\gamma}{r}$$



r = 100 µm P = 14.6 mbar
r = 10 µm P = 146 mbar
r = 1 µm P = 1.46 bar

Another derivation of *p* (based on work)

- A bubble is at mechanical equilibrium.
- An increase of internal pressure must be compensated by an increase of surface energy.

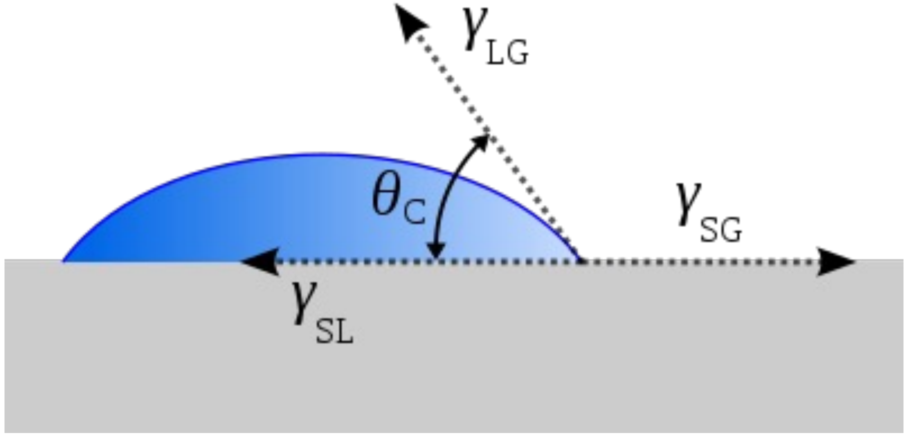
$$\Delta P \cdot dV = \gamma_{lg} \cdot dA$$

$$\Delta P = \frac{2\gamma_{lg}}{r}$$

Surface tension - contact angle (Young-Laplace equ.)

Contact angle between liquid and surface :

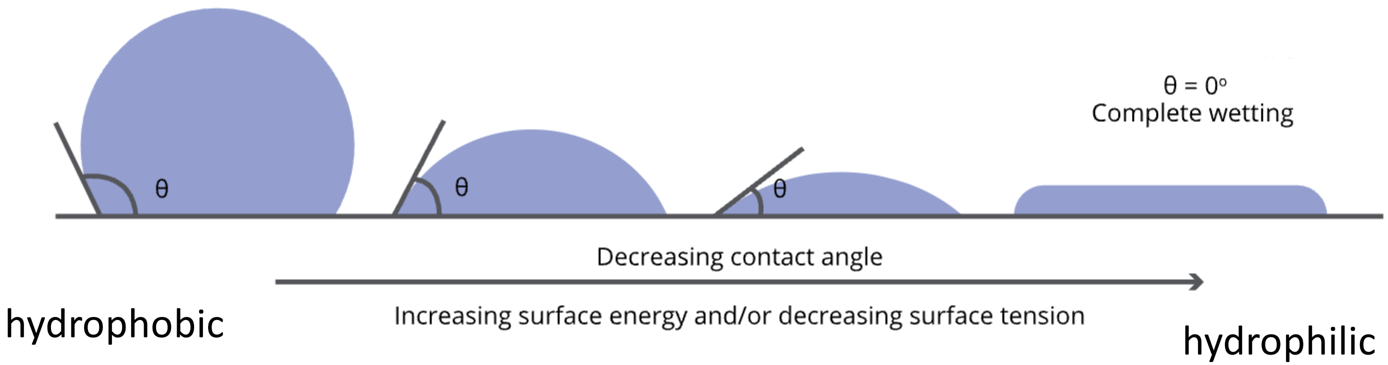
$$\cos\theta = \frac{\gamma_{sg} - \gamma_{sl}}{\gamma_{lg}}$$



polystyrene	water	86°
glass	water	14°
silicone	water	110°

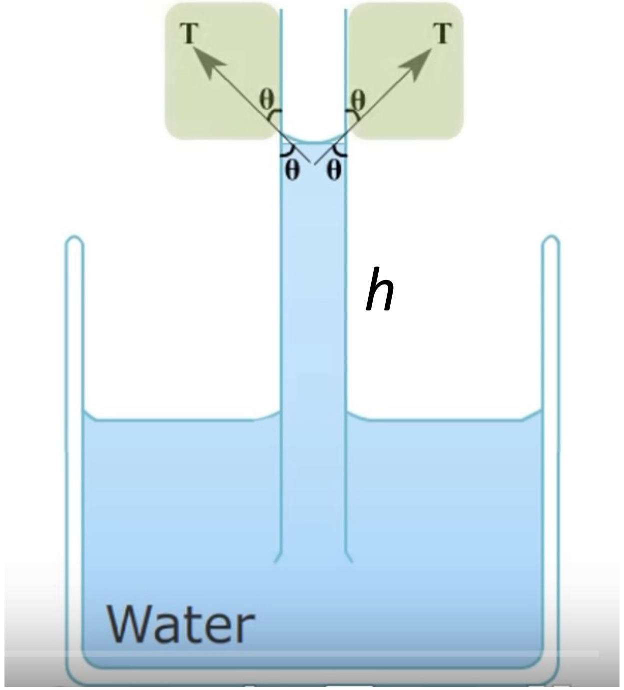
For perfectly wetting surface θ = 0°

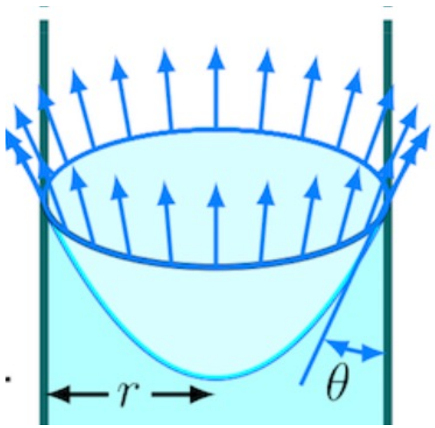
For perfectly non wetting surface θ = 180°



Capillary pressure in a tube

$$\cos\theta = \frac{\gamma_{sg} - \gamma_{sl}}{\gamma_{lg}}$$





$$p = \frac{2\,\gamma}{r}\cos\theta$$

If $\theta = 45°$

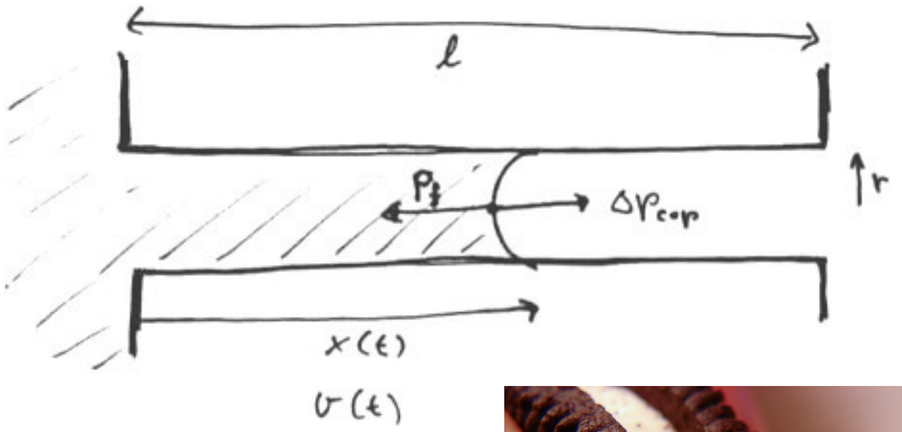
r= 100 µm p = 0.01 bar

r= 10 µm p = 0.1 bar

r= 1 µm p = 1 bar

$$\boxed{h = \frac{2\,\gamma\cos(\theta)}{\rho\; g\; r}}$$

Filling of a capillary – Washburn equation.
Time to self-fill a capillary, ignoring inertia



Capillary pressure in a channel:

$$\Delta p_{cap}(t) = \frac{2\gamma_{gl}\cos\theta}{r}$$

$$\Delta P_{friction}(t) = \frac{\bar{v}\eta x}{8r^2} = \frac{\eta}{8r^2} \cdot x(t)\frac{dx(t)}{dt}$$

Equation of motion: $$x\frac{dx}{dt} = \frac{r \cdot \gamma_{gl}\cos\theta}{\eta}$$

Solution : $$\boxed{\boldsymbol{x(t) = \sqrt{\frac{\gamma r\, t \cos\theta}{2\eta}}}}$$

$$\boxed{x(t) = \sqrt{C\,t} \qquad C = \frac{\gamma\, r \cos\theta}{2\eta}}$$

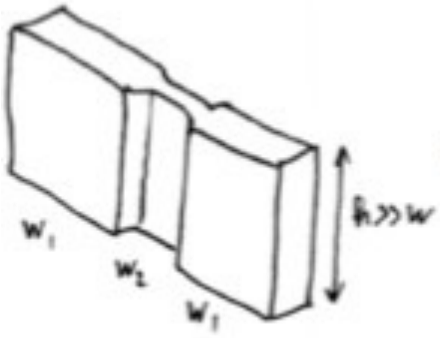
Time to fill a length l_0 $$t_w(l_0) = \frac{\eta}{r \cdot \gamma_{gl}\cos\theta} \cdot l_0^2$$ $$t_w \propto \frac{l^2}{r}$$

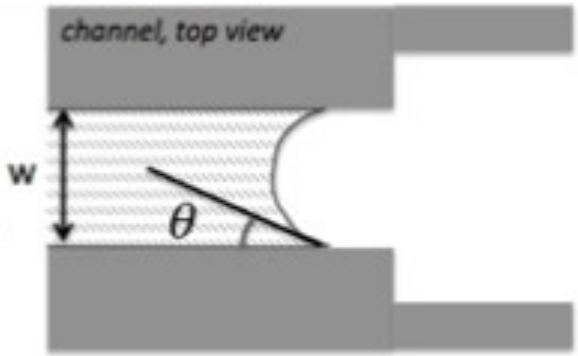
- Does the wetting front goes faster when increasing the radius *r*?

  Yes… but eventually no longer true because of inertial effects !

Capillary stop valve by channel restriction

Here we discuss capillary pressure in hydrophilic rectangular channels where $h >> w$. In this case, the capillary pressure is dominated by the lateral walls:

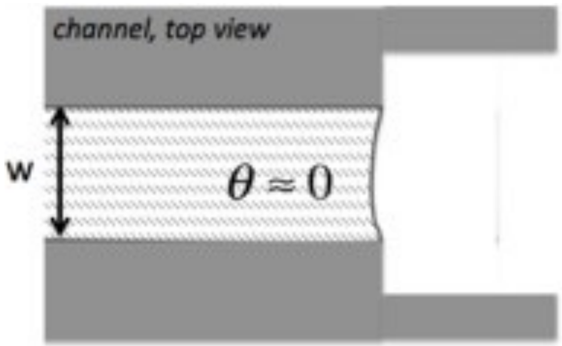


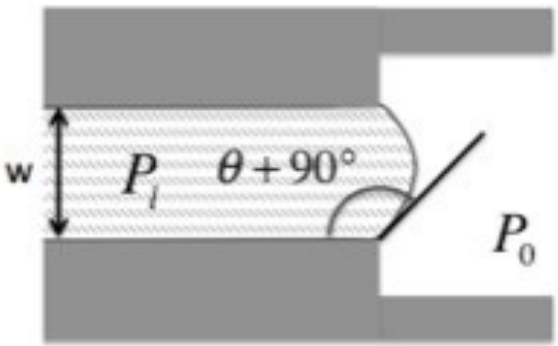


$$\Delta P_{cap} = -2\gamma\left(\frac{\cos\theta_h}{h} + \frac{\cos\theta_w}{w}\right)$$

$$for\ h >> w \quad \Delta P_{cap} = -2\gamma\frac{\cos\theta_h}{w}$$

When the liquid front reaches the **end of the small section**, the "effective" contact angle opens up and thus decreases the capillary pressure. This **stops** the capillary flow. An additional **pressure pulse** is needed to restart the flow, until the contact angle on orthogonal walls is less than 90° (for the case of sharp 90° side walls)





$$\Delta P_{break} = P_0 - P_l = -2\gamma\frac{\cos(\theta + 90)}{w}$$

$$\Delta P_{break} = 2\gamma\frac{\sin\theta}{w}$$

Biosensors **2021**, *11*(10), 405; https://doi.org/10.3390/bios11100405

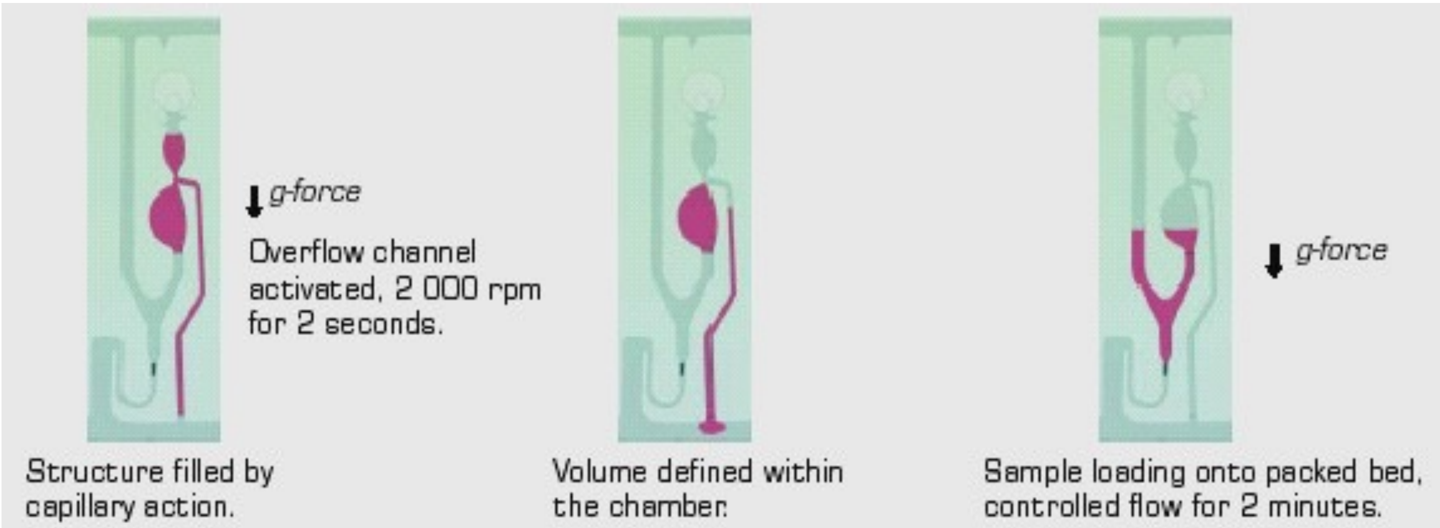
Centrifuge fluidics and capillary barriers

- Use centrifugal force to create pressure.
- Use capillary stops to control progression of liquid in compartments

Equivalent pressure on a liquid plug (length l) $$p_{eq} = \frac{F}{A} = \frac{\rho \cdot l \cdot A\omega^2 R}{A} = \rho \cdot l \cdot \omega^2 \cdot R$$

Displacement speed: $$\rho l \omega^2 R = \frac{\bar{v}\eta l}{8r^2} \quad \Rightarrow \quad \bar{v} = \frac{8\rho\omega^2 R}{\eta} r^2 \qquad v \propto r^2$$

- The plug speed is <u>independent</u> of its <u>length</u> (not as in pressure driven flow)

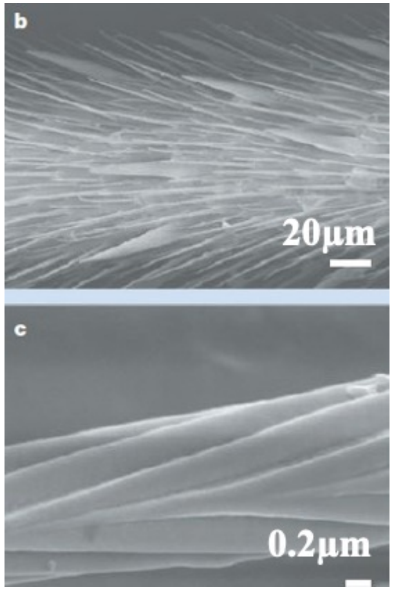


<u>Comparison</u>: pressure driven plug: $$\Delta p = \frac{\bar{v}\eta l}{8r^2} \quad \Rightarrow \quad \bar{v} = \frac{8\Delta p}{\eta}\frac{r^2}{l}$$

Water striders (Gerris remigis)

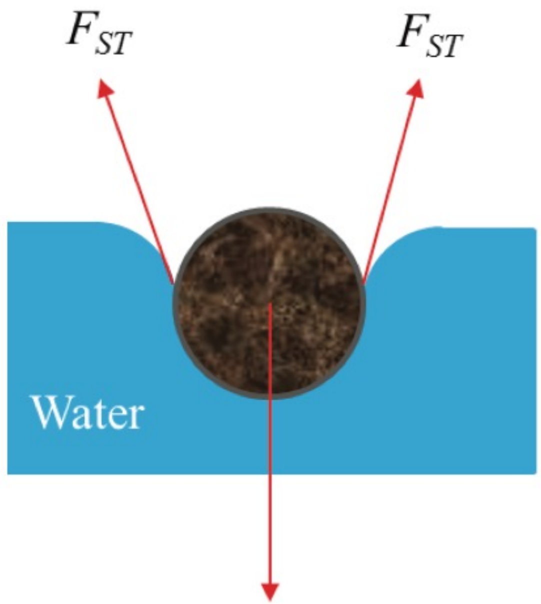
Surface tension – hydrophobic object "resting on water"

Water striders (Gerris remigis)



A single leg can support 15 times the weight of the insect

$$F_{St} = \gamma\, 2\pi\, r$$



Floatability criterion ($f > 1$ to float):

$$f = \frac{F_{st}}{F_{grav}}$$

$$f \propto r^{-2} \propto L^{-2}$$

$$r_{crit} = \sqrt{\frac{3}{2}\frac{\gamma}{\Delta\rho g}}$$

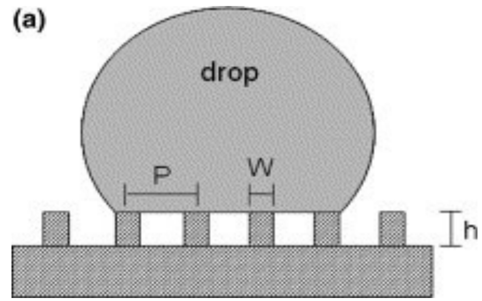
$$F_{grav} = \frac{4}{3}\pi r^3 \Delta\rho\, g$$

= 2 mm for glass bead on water

Lotus leaf effect (super-hydrophobic) effective contact angle

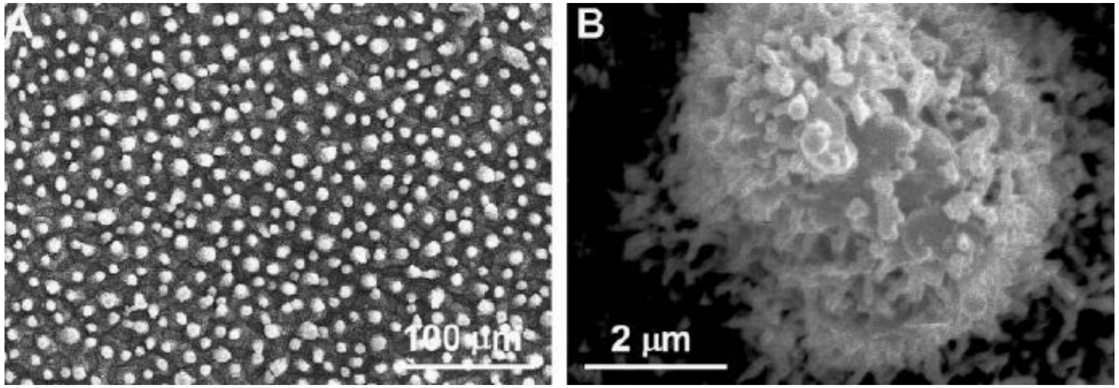
$$\cos\theta_c = \sigma_1 \cos\theta_1 + \sigma_2 \cos\theta_2$$

$$\cos\theta_{cb} = \sigma_1 \cos\theta_1 - \sigma_2$$



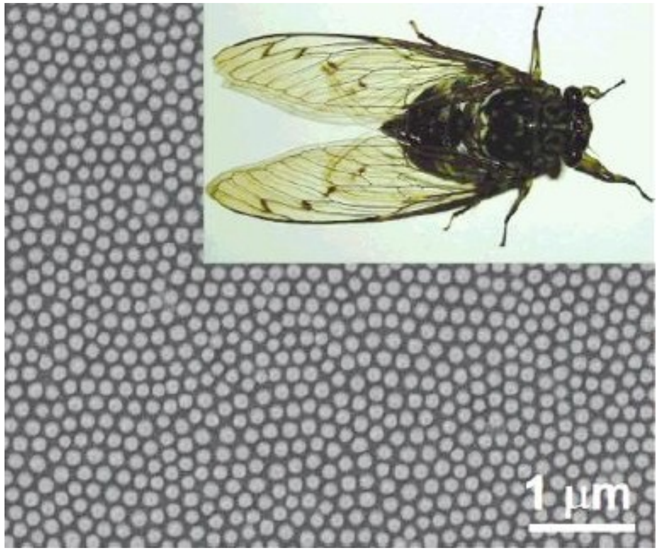
Very high local curvature

Superhydrophobic behaviour is due to surface corrugation:

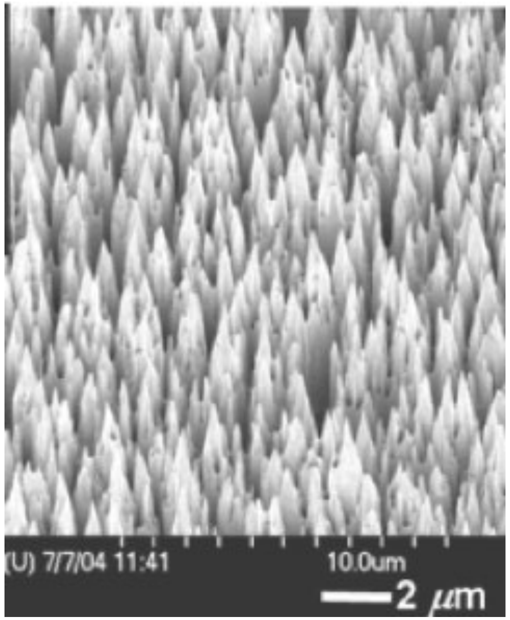


Self-cleaning effect :

Superhydrophobic surfaces



Cicada fly (Cicada orni)



Sun et al., Acc. Chem Res. 2005, 38, 644

Rebouncing of falling droplets on superhydrophobic surface:

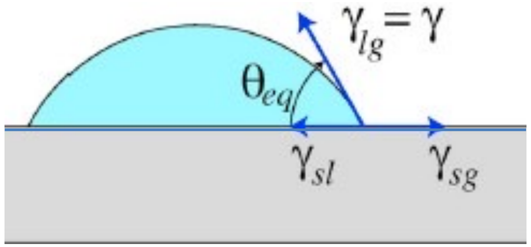
https://doi.org/10.1021/la8003504

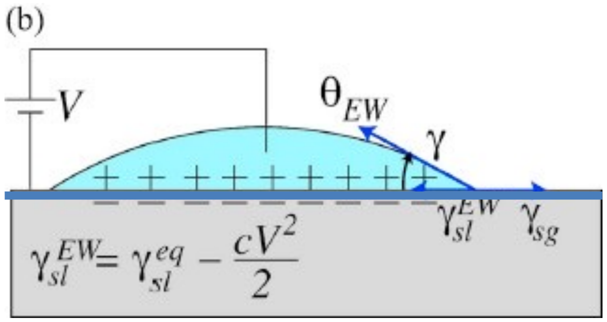
Electrowetting at surfaces

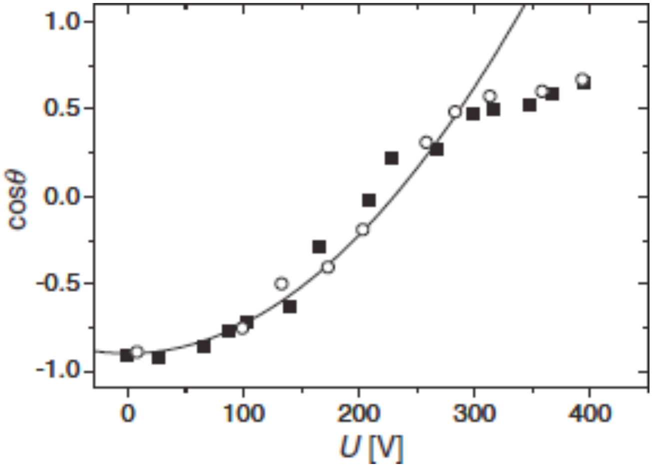
The contact angle can be modified by electrostatic surface energy. The additional electrostatic surface energy is given by the capacitance energy $\frac{1}{2}cV^2$

where c is the capacitance per unit area $c = \frac{C}{A} = \frac{\varepsilon_0 \varepsilon_r}{d}$

ε_r, d dielectric constant and thickness of the insulator
V voltage on the electrode







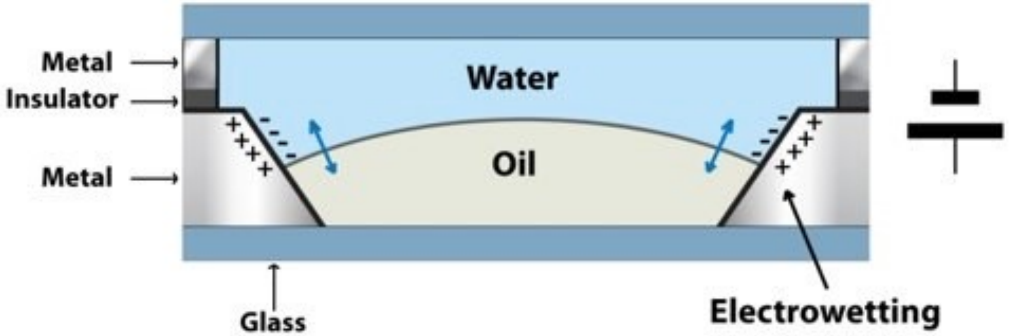
$$\gamma_{sg} - \gamma_{sl} - \gamma \cos\theta = 0$$

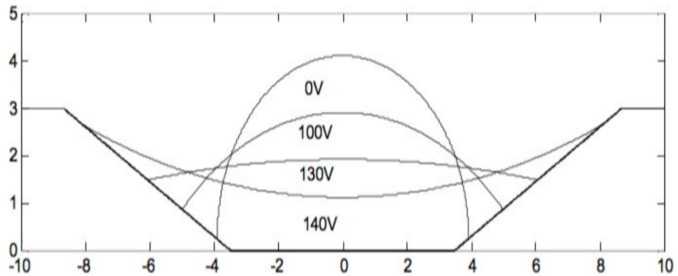
$$\gamma_{sg} - \left(\gamma_{sl} - \frac{1}{2}cV^2\right) - \gamma \cos\theta = 0$$

Electrowetting optics

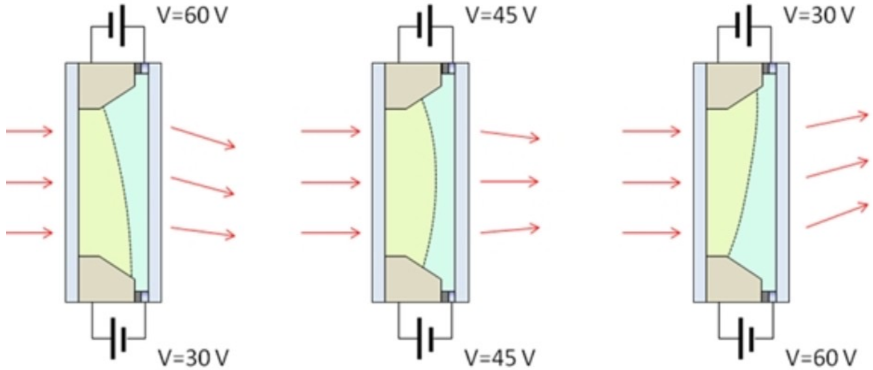
Idea: a liquid drop forms a refractive lens
The the radius of curvature can be changed by electrowetting on the contact line of the liquid
In practice, the interface is made between water and oil

Liquid lens for autofocus:



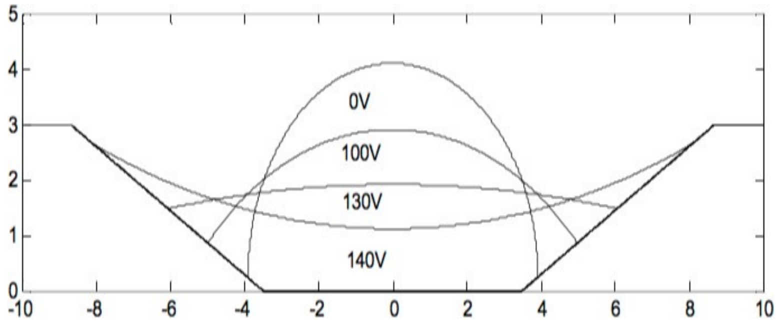


Liquid lens for image stabilitzation:



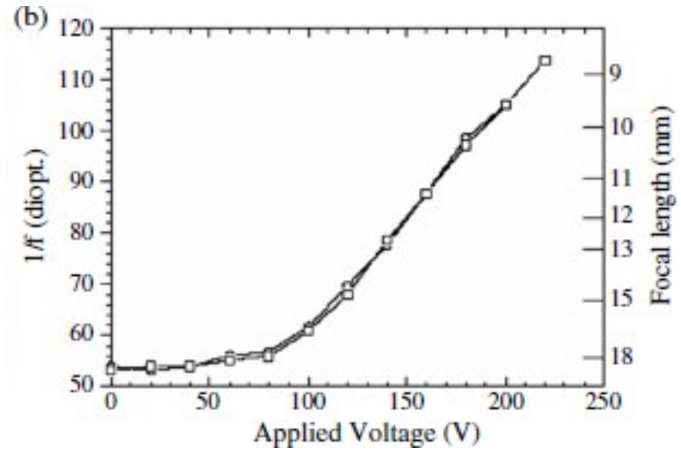
B. Berge, J. Peseux, "Variable focal lens controlled by an external voltage: an application of electrowetting", Eur. Phys. J. E. 3, pp159-163, 2000.
www.varioptic.com

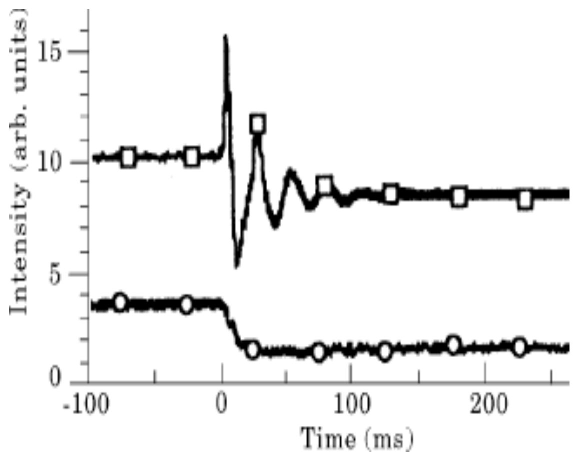
Electrowetting optics



B. Berge, J. Peseux, “Variable focal lens controlled by an external voltage: an application of electrowetting”, Eur. Phys. J. E. 3, pp159-163, 2000.
www.varioptic.com (now Corning)

Liquid lens





- The contact angle is proportional to V^2, but due to the shape of the electrode, the optical power is almost linear with voltage in the range between 100 V and 250 V
- Because of small size: fast response. Damping can be increased by adding polymer molecules in water.

Scaling: how large a liquid lens can be ?

The capillary length is a characteristic length scale for an interface between two fluids which is subject to gravitational acceleration and to surface tension.

It is defined as :

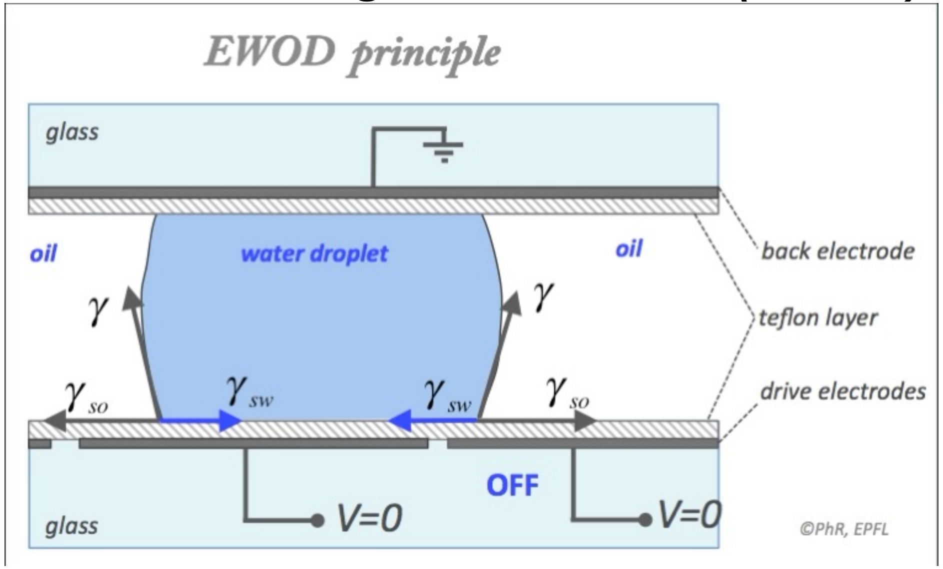
$$\lambda_c = \sqrt{\frac{\gamma}{\rho g}}$$

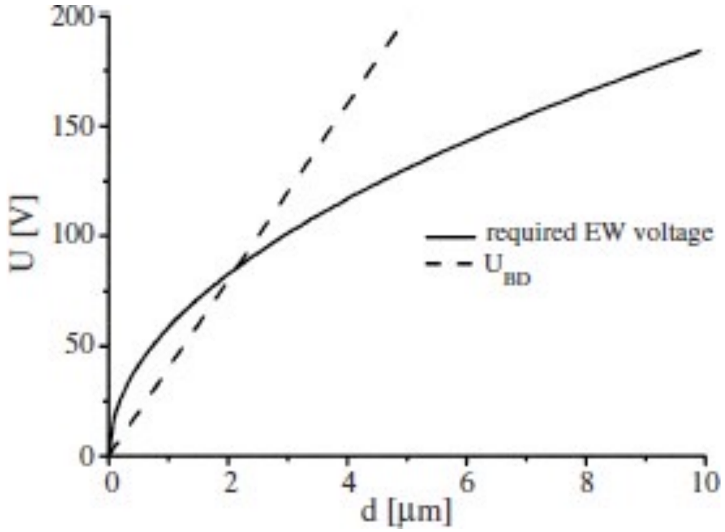
γ: surface tension of interface between fluids

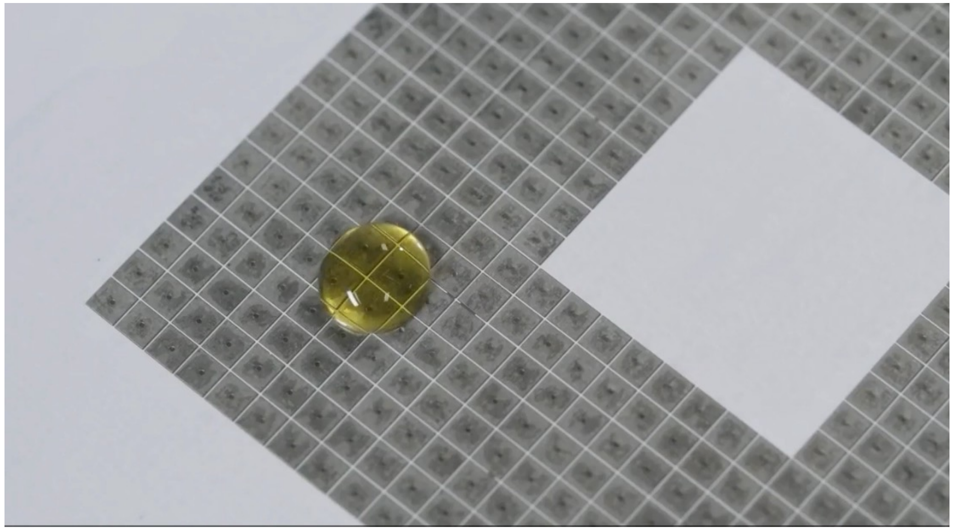
ρ: density, g: gravity acceleration

For water-air interface, the capillary length is around 2.7 mm.
Sessile drops with a radius smaller than the capillary length are almost spherical.
In liquid lens, the capillary length is much increased by choosing **two liquids** with **same density**

Electrowetting on dielectrics (EWOD)



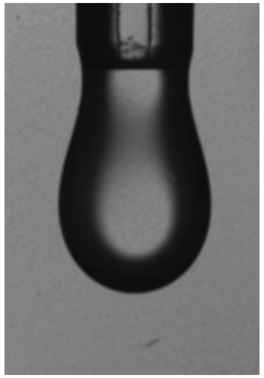
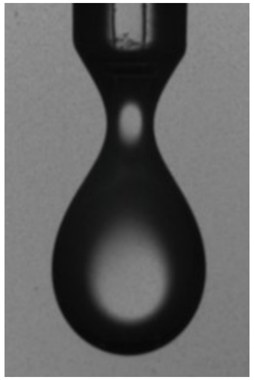
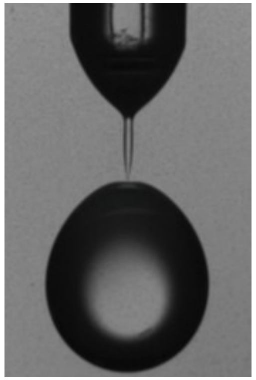




A good review on electrowetting:

F. Mugele et al., J. Phys:condens. Mater 17 (2005) R705-R774

The hanging drop

The force due to surface tension is proportional to the length of the boundary between the liquid and the tube = equal the circumference. The force due to surface tension is given by :

$$F_\gamma = \pi d \gamma$$

Gravity force : $F_g \propto \frac{4}{3}\pi r^3 \cdot \rho g$

The vertical component of the capillary force is equal to the weight of the drop:

$mg = \pi d\gamma \cdot \sin\alpha$ where α is the angle of contact with the tube,

At the limit: $mg = \pi d\gamma$ or $\rho \frac{4}{3}\pi r^3 g = \pi d\gamma$ $r \propto d^{1/3}$

It is difficult to make very small drops by simple gravity detachment because of capillary pressure increase.

<u>Ideas</u> : - drop ejection for pressure pulse => drop-on-demand dispensers, inkjet
- using shear flow => droplets microfluidic

The pitch drop experiment (goutte de poix)

- Extremely viscous fluid. One drop falls every 8 to 10 years !
- Started at Queensland University in 1927
- Measured viscosity of pitch: 1012 time that of water

Date	Event	Years
1927	Hot pitch poured	
October 1930	Stem cut	
December 1938	1st drop fell	8.1
February 1947	2nd drop fell	8.2
April 1954	3rd drop fell	7.2
May 1962	4th drop fell	8.1
August 1970	5th drop fell	8.3
April 1979	6th drop fell	8.7
July 1988	7th drop fell	9.2
November 2000	8th drop fell	12.3
April 2014	9th drop	13.4

webcam: http://www.thetenthwatch.com

Drop formation by Rayleigh–Plateau instability

Plateau-Rayleigh instability is a physics phenomenon where a column of liquid breaks up into droplets in order to minimize it's surface energy.

The breakup distance depends on the speed of ejection and is roughly proportional to $r^{3/2}$

Drop formation by viscous shear forces (in biphasic flows)

When a fluid A arrives from a perpendicular branch in a main channel, droplets of this fluid can be detached if a fluid B is flowing. The size is given by a balance between the capillary pressure necessary to form a droplet and the viscous stress from the fluid B flowing in a channel.

The Capillary number is defined as:

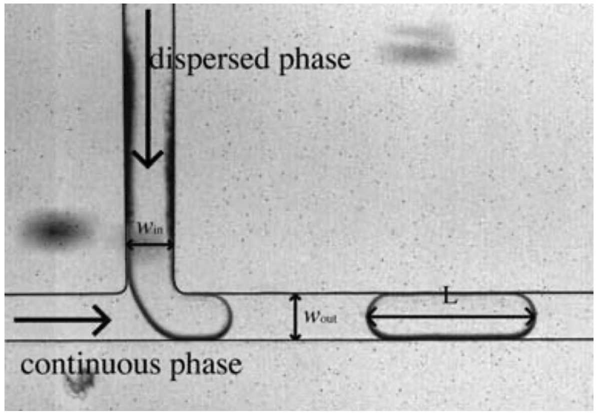
$$C_A = \frac{\eta \cdot v}{\gamma}$$

High capillary number: behavior is dominated by shear viscous forces
Small capillary number: behavior is dominated by surface tension

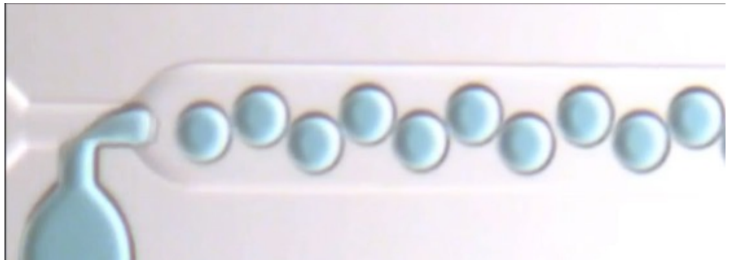
Droplets formation by breakup

Microfluidic confinement allows the creation of **small** droplets with, very **homogeneous** size distribution

Water in oil droplets require hydrophobic walls and a surfactant



$$\frac{L}{a} \simeq 1 + \frac{Q_w}{Q_{oil}}$$



Berthier, p. 175

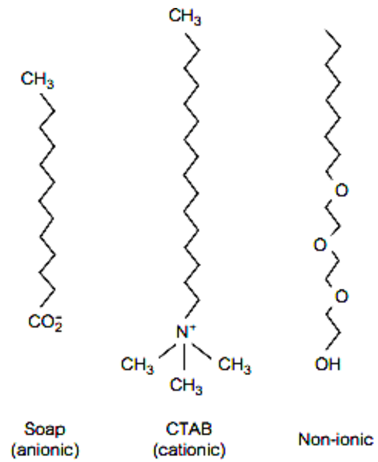
Surfactants:

The stability of droplets is maintained by **surfactants**.

Molecules made of two parts:

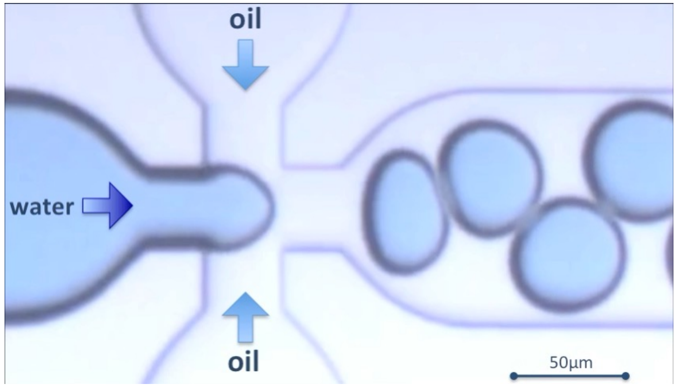
hydrophilic part 'likes' water
hydrophobic part does not like water

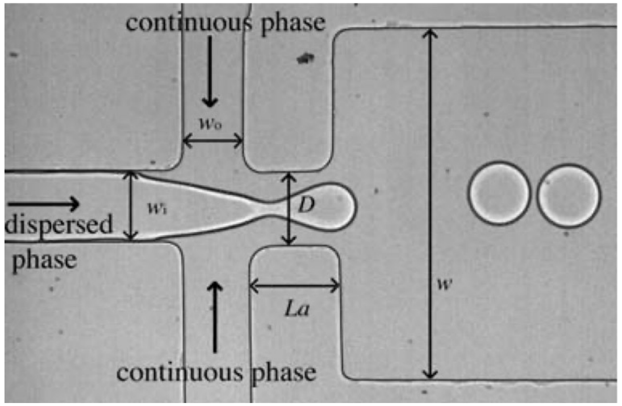
Tabeling, p.110

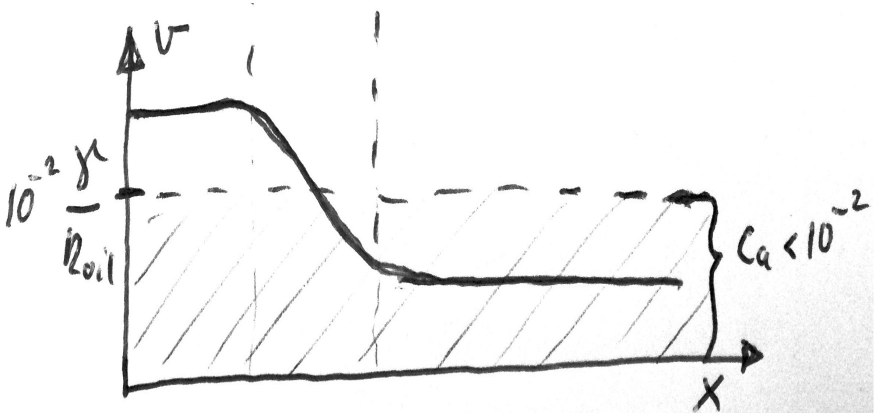


Droplets – formation by capillary instability

The droplets are formed by **breaking** the jet by capillary forces (surface tension).







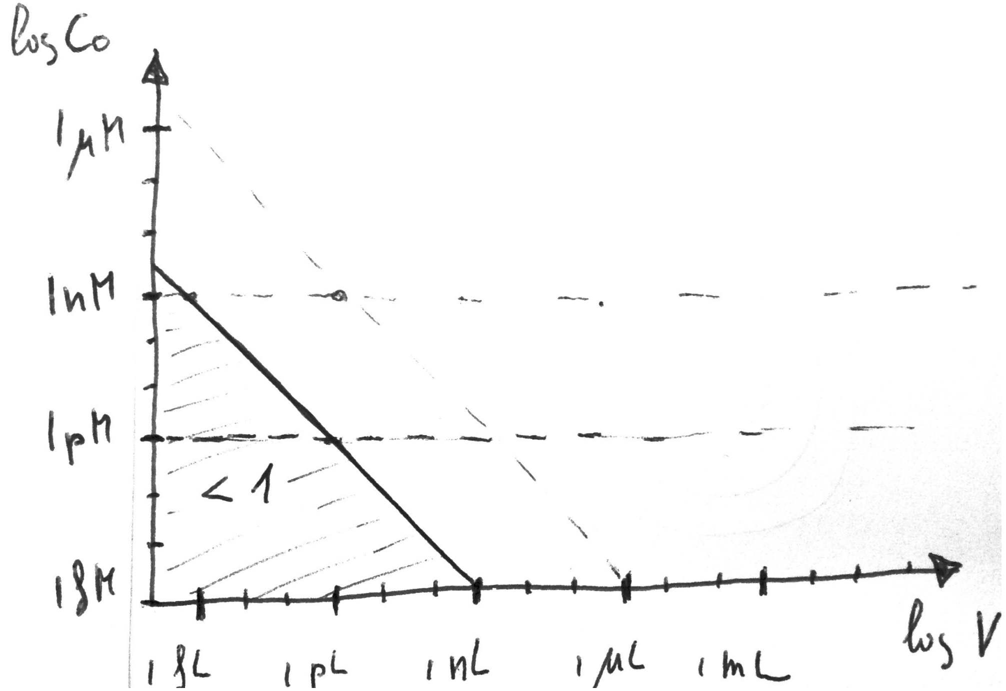
$$Ca_{crit} = \frac{\eta_{oil}\, v}{\gamma} < 10^{-2}$$

The jet is broken when Ca number is below 10^{-2} *(η : viscosity of continuous phase , γ : surface tension)*
The Capillary number represents the ratio of **viscous** forces to **surfaces** forces.

Gunther and Jensen, Lab Chip, 2006, 6, 1487–1503 and Berthier, p. 183

The dilution (homeopathic) limit

In very small droplet volume, we reach conditions where less than one molecule is present per droplet



Conc.:	*Nb of molec/mL*
1M	n= 6.10^{20} molec/mL
1mM	n= 6.10^{17} molec/mL
1μM	n= 6.10^{14} molec/mL
1nM	n= 6.10^{11} molec/mL
1pM	n= 6.10^{8} molec/mL
1fM	n= 6.10^{5} molec/mL

For 1nM concentration :

Volume	1μL	1nL	1pL	1 fL
Size (cube)	1mm	100μm	10μm	1μm
Nb of molecules	6.10^{8}	6.10^{5}	600	0.6

Concentration: *1 M = 1 Mol/liter* *1 Mol = 6.10^{23} molecules*

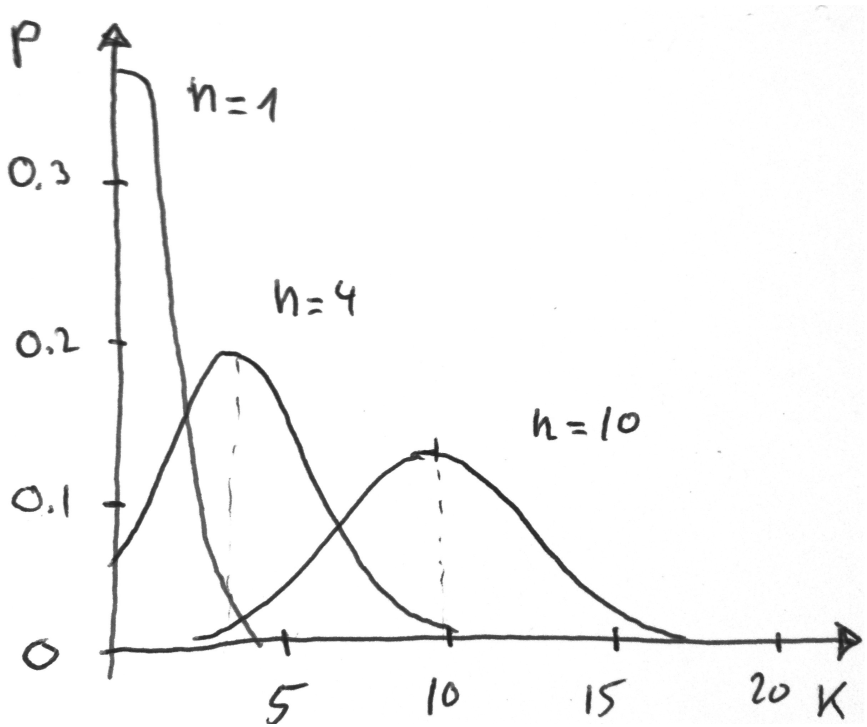
Occupancy statistics at low concentrations in small volumes

The probability distribution of molecules in the well obeys **Poisson** law:

$$p(k;n) = \frac{n^k \cdot e^{-n}}{k!}$$

n: *average number of molecules in the volume* $(n = C_0 \cdot V \cdot 1000 \cdot N_A)$

p(k): *probability of having k molecules in a volume*



Probability distribution for different average numbers n

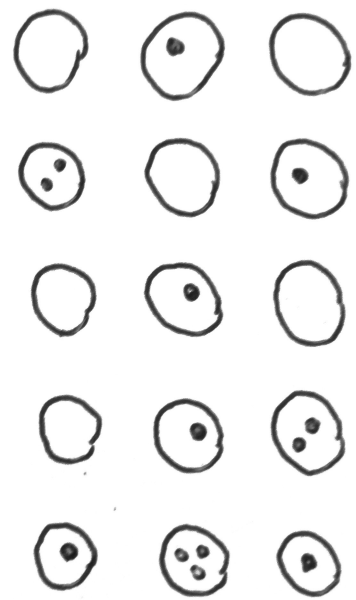
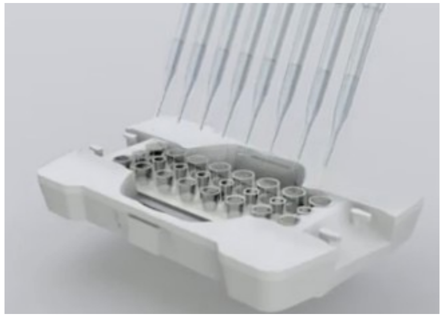
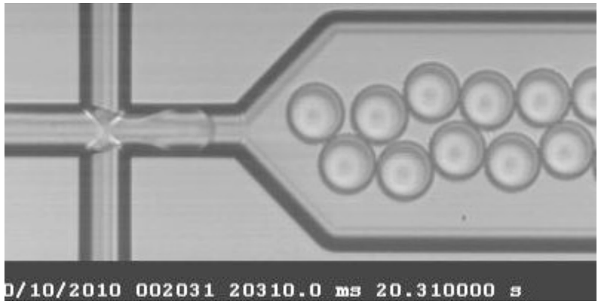


Illustration for for n=1

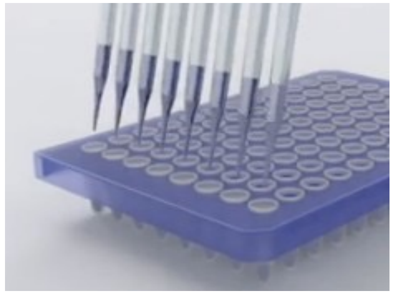
Digital PCR in droplet format (example: Bio-rad, ex QuantaLife)



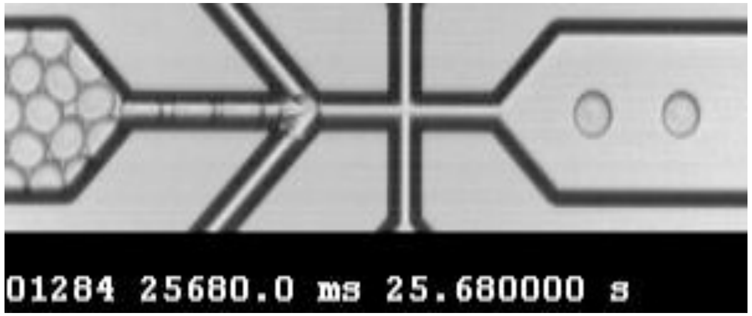
(1) Load samples in chip



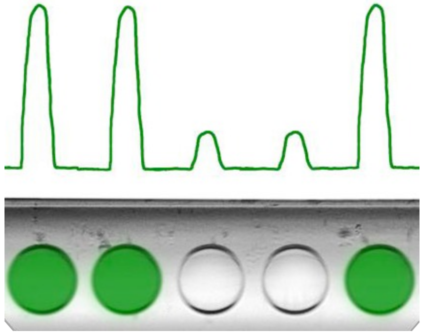
(2) **Partition** into droplets



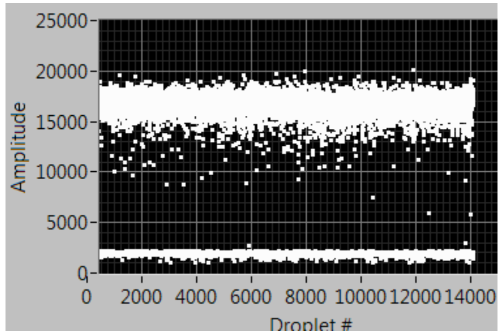
(3) **off-chip** PCR cycles



(4) **Reload** in chip and **align** the droplets



(5) **Read** in flow



(6) Analysis

Measured probability of **zeros**

$$p_- = (1 - p_+) = 1 - \frac{2'000}{20'000} = 0.1$$

Number of molecules per chamber

$$n = -\ln(1 - p_+) = 0.10535$$

Calculated concentration of DNA

$$C = \frac{n}{V_0} = \frac{0.10535}{10^{-9} \cdot 1000 \cdot N_A}$$